On the asymptotic cardinal function of the multiquadric $\varphi(r) = (r^2 + c^2)^{1/2}$ as $c \to \infty$

B. J. C. Baxter
Department of Applied Mathematics and Theoretical Physics,
Silver Street, Cambridge CB3 9EW,
England

A radial basis function approximation has the form

$$s(x) = \sum_{j \in \mathcal{Z}^d} y_j \, \varphi(\|x - x_j\|_2), \qquad x \in \mathcal{R}^d,$$

where $\varphi\colon [0, \infty) \to \mathcal{R}$ is some given function, $(y_j)_{j \in \mathcal{Z}^d}$ are real coefficients, and the *centres* $(x_j)_{j \in \mathcal{Z}^d}$ are points in $\mathcal{R}^d$. It is known that radial basis function approximations using the multiquadric $\varphi(r) = (r^2 + c^2)^{1/2}$ possess many useful and interesting properties when the centres form an infinite regular lattice. We analyse the limiting case as $c \to \infty$ and identify a class of functions that arise as uniform limits of the multiquadric interpolants. In the univariate case, we observe that the cardinal function for the multiquadric becomes the sinc function as $c \to \infty$. The limit of the multivariate cardinal function is also identified.

1. Introduction

The radial basis function approach to interpolating a function $f\colon \mathcal{R}^d \to \mathcal{R}$ on the integer lattice $\mathcal{Z}^d$ is as follows. Given a continuous univariate function $\varphi\colon [0,\infty) \to \mathcal{R}$, we seek a *cardinal function*

$$\chi(x) = \sum_{j\in\mathcal{Z}^d} a_j \varphi(\|x-j\|), \qquad x \in \mathcal{R}^d, \tag{1.1}$$

that satisfies

$$\chi(k) = \delta_{0,k}, \qquad k \in \mathcal{Z}^d.$$

Therefore

$$If(x) = \sum_{j\in\mathcal{Z}^d} f(j)\chi(x-j), \qquad x \in \mathcal{R}^d, \tag{1.2}$$

is an interpolant to f on the integer lattice whenever (1.2) is well defined. Here $\|\cdot\|$ is the Euclidean norm on $\mathcal{R}^d$. This approach provides a useful and flexible family of approximants for many choices of φ, but here we concentrate on the Hardy multiquadric $\varphi_c(r) = (r^2+c^2)^{1/2}$. For this function, Buhmann (1990) has shown that a cardinal function χ_c exists and its Fourier tranform is given by the equation

$$\hat\chi_c(\xi) = \frac{\hat\varphi_c(\|\xi\|)}{\sum_{k\in\mathcal{Z}^d}\hat\varphi_c(\|\xi+2\pi k\|)}, \qquad \xi \in \mathcal{R}^d, \tag{1.3}$$

where $\{\hat\varphi_c(\|\xi\|) : \xi \in \mathcal{R}^d\}$ is the generalized Fourier transform of $\{\varphi_c(\|x\|) : x \in \mathcal{R}^d\}$. Further, χ_c possesses a classical Fourier transform (see Jones (1982) or Schwartz (1966)). In this paper, we prove that $\hat\chi_c$ enjoys the following property:

$$\lim_{c\to\infty} \hat\chi_c(\xi) = \begin{cases} 1, & \xi \in (-\pi,\pi)^d, \\ 0, & \xi \notin [-\pi,\pi]^d, \end{cases} \tag{1.4}$$

which sheds new light on the approximation properties of the multiquadric as $c \to \infty$. For example, in the case $d = 1$, (1.4) implies that $\lim_{c\to\infty}\chi_c(x) = \mathrm{sinc}(x)$, providing a perhaps unexpected link with sampling theory and the classical theory of the Whittaker cardinal spline. Further, our work has links with the error analysis of Buhmann and Dyn (1991) and illuminates the explicit calculation of Section 4 of Powell (1991). It may also be compared with the results of Madych and Nelson (1990) and Madych (1990), because these papers present analogous results for polyharmonic cardinal splines.

2. Some properties of the multiquadric

The generalized Fourier transform of φ_c is given by

$$\hat\varphi_c(\|\xi\|) = -\pi^{-1}(2\pi c/\|\xi\|)^{(d+1)/2} K_{(d+1)/2}(c\|\xi\|), \tag{2.1}$$

for nonzero $\xi \in \mathcal{R}^d$ (see Jones (1982)). Here $\{K_\nu(r) : r > 0\}$ are the modified Bessel functions, which are positive and smooth in $\mathcal{R}^+$, have a pole at the origin, and decay exponentially (see Abramowitz and Stegun (1970)). There is an integral representation for these modified Bessel functions (Abramowitz and Stegun (1970), equation 9.6.23) which transforms (2.1) into a highly useful formula for $\hat{\varphi}_c$:

$$\hat{\varphi}_c(\|\xi\|) = -\lambda_d c^{d+1} \int_1^\infty \exp(-cx\|\xi\|)(x^2-1)^{d/2}\, dx, \tag{2.2}$$

where $\lambda_d = \pi^{d/2}/\Gamma(1+d/2)$. A simple consequence of (2.2) is the following lemma, which bounds the exponential decay of $\hat{\varphi}_c$.

Lemma 2.1. *If* $\|\xi\| > \|\eta\| > 0$*, then*

$$|\hat{\varphi}_c(\|\xi\|)| \le \exp[-c(\|\xi\| - \|\eta\|)]\, |\hat{\varphi}_c(\|\eta\|)|.$$

Proof. Applying (2.2), we obtain

$$\begin{aligned}|\hat{\varphi}_c(\|\xi\|)| &= \lambda_d c^{d+1} \int_1^\infty \exp[-cx(\|\xi\| - \|\eta\|)] \exp(-cx\|\eta\|)\, (x^2-1)^{d/2}\, dx \\ &\le \exp(-c(\|\xi\| - \|\eta\|))\, |\hat{\varphi}_c(\|\eta\|)|,\end{aligned}$$

providing the desired bound. ■

We now prove our main result. We let $I\colon \mathcal{R}^d \to \mathcal{R}$ be the *characteristic function* of the cube $[-\pi,\pi]^d$, that is

$$I(\xi) = \begin{cases} 1, & \xi \in [-\pi,\pi]^d, \\ 0, & \xi \notin [-\pi,\pi]^d. \end{cases}$$

Proposition 2.2. *Let* ξ *be any fixed point of* $\mathcal{R}^d$*. We have*

$$\lim_{c\to\infty} \hat{\chi}_c(\xi) = I(\xi),$$

if $\|\xi\|_\infty \ne \pi$*, that is* ξ *does not lie in the boundary of* $[-\pi,\pi]^d$*.*

Proof. First, suppose that $\xi \notin [-\pi,\pi]^d$. Then there exists a nonzero integer k_0 such that $\|\xi + 2\pi k_0\| < \|\xi\|$, and Lemma 2.1 provides the bounds

$$\begin{aligned}|\hat{\varphi}_c(\|\xi\|)| &\le \exp[-c(\|\xi\| - \|\xi + 2\pi k_0\|)] |\hat{\varphi}_c(\|\xi + 2\pi k_0\|)| \\ &\le \exp[-c(\|\xi\| - \|\xi + 2\pi k_0\|)] \sum_{k\in\mathcal{Z}^d} |\hat{\varphi}_c(\|\xi + 2\pi k\|)|.\end{aligned}$$

Thus, applying (1.3) and remembering that $\hat{\varphi}_c$ does not change sign, we have

$$0 \le \hat{\chi}_c(\xi) \le \exp[-c(\|\xi\| - \|\xi + 2\pi k_0\|)], \qquad \xi \notin [-\pi,\pi]^d. \tag{2.3}$$

The upper bound of (2.3) converges to zero as $c \to \infty$, which completes the proof for this range of ξ.

Suppose now that $\xi \in (-\pi, \pi)^d$. Further, we shall assume that ξ is nonzero, because we know that $\hat{\chi}_c(0) = 1$ for all values of c. Then $\|\xi + 2\pi k\| > \|\xi\|$, for every nonzero integer $k \in \mathcal{Z}^d$. Now (1.3) provides the expression

$$\hat{\chi}_c(\xi) = \Big(1 + \sum_{k \in \mathcal{Z}^d \setminus \{0\}} \Big| \frac{\hat{\varphi}_c(\|\xi + 2\pi k\|)}{\hat{\varphi}_c(\|\xi\|)} \Big| \Big)^{-1}. \tag{2.4}$$

We shall show that

$$\lim_{c \to \infty} \sum_{k \in \mathcal{Z}^d \setminus \{0\}} \left| \frac{\hat{\varphi}_c(\|\xi + 2\pi k\|)}{\hat{\varphi}_c(\|\xi\|)} \right| = 0, \qquad \xi \in (-\pi, \pi)^d, \tag{2.5}$$

which, together with (2.4), implies that $\lim_{c \to \infty} \hat{\chi}_c(\xi) = 1$.

Now Lemma 2.1 implies that

$$\sum_{k \in \mathcal{Z}^d \setminus \{0\}} \left| \frac{\hat{\varphi}_c(\|\xi + 2\pi k\|)}{\hat{\varphi}_c(\|\xi\|)} \right| \le \sum_{k \in \mathcal{Z}^d \setminus \{0\}} \exp[-c(\|\xi + 2\pi k\| - \|\xi\|)], \tag{2.6}$$

and each term of the series on the right converges to zero as $c \to \infty$, since $\|\xi + 2\pi k\| > \|\xi\|$ for every nonzero integer k. Therefore we need only deal with the tail of the series. Specifically, we derive the equation

$$\lim_{c \to \infty} \sum_{\|k\| \ge 2\|e\|} \exp[-c(\|\xi + 2\pi k\| - \|\xi\|)] = 0, \tag{2.7}$$

where $e = [1, 1, \ldots, 1]^T$. Now, if $\|k\| \ge 2\|e\|$, then

$$\|\xi + 2\pi k\| - \|\xi\| \ge 2\pi(\|k\| - \|e\|) \ge \pi\|k\|,$$

remembering that we have $\|\xi\| \le \pi\|e\|$. Hence

$$\sum_{\|k\| \ge 2\|e\|} \exp[-c(\|\xi + 2\pi k\| - \|\xi\|)] \le \sum_{\|k\| \ge 2\|e\|} \exp(-\pi c\|k\|). \tag{2.8}$$

It is a simple exercise to prove that the series $\sum_{\|k\| \ge 2\|e\|} \exp(-\pi\|k\|)$ is convergent. Therefore, given any $\epsilon > 0$, there exists a positive number $R \ge 1$ such that

$$\sum_{\|k\| \ge 2R\|e\|} \exp(-\pi\|k\|) \le \epsilon.$$

Consequently, when $c \ge \lceil R \rceil$ we have the inequality

$$\sum_{\|k\| \ge 2\|e\|} \exp(-\pi c\|k\|) \le \sum_{\|k\| \ge 2R\|e\|} \exp(-\pi\|k\|) \le \epsilon,$$

which establishes (2.5). The proof is complete. ■

3. Multiquadrics and entire functions of exponential type π

Definition 3.1. Let $f \in L^2(\mathcal{R}^d)$. We shall say that f is a *function of exponential type* A if its Fourier transform $\hat{f}$ is supported by the cube $[-A, A]^d$. We shall denote the set of all functions of exponential type A by $E_A(\mathcal{R}^d)$.

We remark that the Paley-Wiener theorem implies that f may be extended to an entire function on $\mathcal{C}^d$ satisfying a certain growth condition at infinity (see Stein and Weiss (1971), pages 108ff). However, we do not use the Paley-Wiener theorem in this paper.

Lemma 3.2. *Let $f \in E_\pi(\mathcal{R}^d) \cap L^2(\mathcal{R}^d)$ be a continuous function. Then we have the equation*

$$\sum_{k\in\mathcal{Z}^d} \hat{f}(\xi + 2\pi k) = \sum_{k\in\mathcal{Z}^d} f(k)\exp(-ik\xi), \tag{3.1}$$

the second series being convergent in $L^2(\mathcal{R}^d)$.

Proof. Let

$$g(\xi) = \sum_{k\in\mathcal{Z}^d} \hat{f}(\xi + 2\pi k), \qquad \xi \in \mathcal{R}^d.$$

At any point $\xi \in \mathcal{R}^d$, this series contains at most one nonzero term, because of the condition on the support of $\hat{f}$. Hence g is well defined. Further, we have the relations

$$\int_{[-\pi,\pi]^d} |g(\xi)|^2 \, d\xi = \int_{\mathcal{R}^d} |\hat{f}(\xi)|^2 \, d\xi < \infty,$$

since the Parseval theorem implies that $\hat{f}$ is an element of $L^2(\mathcal{R}^d)$. Thus $g \in L^2([-\pi,\pi]^d)$ and its Fourier series

$$g(\xi) = \sum_{k\in\mathcal{Z}^d} g_k \exp(ik\xi),$$

is convergent in $L^2([-\pi,\pi]^d)$. The Fourier coefficients are given by the expressions

$$g_k = (2\pi)^{-d}\int_{[-\pi,\pi]^d} \hat{f}(\xi)\exp(-ik\xi)\, d\xi = (2\pi)^{-d}\int_{\mathcal{R}^d} \hat{f}(\xi)\exp(-ik\xi)\, d\xi = f(-k),$$

where the final equation uses the Fourier inversion theorem for $L^2(\mathcal{R}^d)$. The proof is complete. ■

We observe that an immediate consequence of the lemma is the convergence of the series $\sum_{k\in\mathcal{Z}^d}[f(k)]^2$, by the Parseval theorem.

For the following results, we shall need the fact that $\chi_c \in L^2(\mathcal{R}^d)$, which is a consequence of the analysis of Buhmann (1990).

Lemma 3.3. *Let $f \in E_\pi(\mathcal{R}^d) \cap L^2(\mathcal{R}^d)$ be a continuous function. For each positive integer n, we define the function*

$$\widehat{S_c^n f}(\xi) = \left(\sum_{\|k\|_1 \le n} f(k) \exp(-ik\xi) \right) \hat{\chi}_c(\xi), \qquad \xi \in \mathcal{R}^d. \tag{3.2}$$

Then $\{S_c^n f : n = 1, 2, \ldots\}$ forms a Cauchy sequence in $L^2(\mathcal{R}^d)$.

Proof. Let $Q_n : \mathcal{R}^d \to \mathcal{R}$ be the trigonometric polynomial

$$Q_n(\xi) = \sum_{\|k\|_1 \le n} f(k) \exp(-ik\xi), \tag{3.3}$$

so that $\widehat{S_c^n f}(\xi) = Q_n(\xi)\hat{\chi}_c(\xi)$. It is a consequence of Lemma 3.2 that this sequence of functions forms a Cauchy sequence in $L^2([-\pi, \pi]^d)$. Indeed, we shall prove that for $m \ge n$ we have

$$\|\widehat{S_c^m f} - \widehat{S_c^n f}\|_{L^2(\mathcal{R}^d)} \le \|Q_m - Q_n\|_{L^2([-\pi,\pi]^d)}, \tag{3.4}$$

so that the sequence of functions $\{\widehat{S_c^n f} : n = 1, 2, \ldots\}$ is a Cauchy sequence in $L^2(\mathcal{R}^d)$.

Now Fubini's theorem provides the relation

$$\begin{aligned} \|\widehat{S_c^m f} - \widehat{S_c^n f}\|^2_{L^2(\mathcal{R}^d)} &= \int_{\mathcal{R}^d} |Q_m(\xi) - Q_n(\xi)|^2 \, \hat{\chi}_c^2(\xi) \, d\xi \\ &= \int_{[-\pi,\pi]^d} |Q_m(\xi) - Q_n(\xi)|^2 \left(\sum_{l \in \mathcal{Z}^d} \hat{\chi}_c^2(\xi + 2\pi l) \right) d\xi. \end{aligned} \tag{3.5}$$

However, (1.3) gives the bound

$$\begin{aligned} \sum_{l \in \mathcal{Z}^d} \hat{\chi}_c^2(\xi + 2\pi l) &= \sum_{l \in \mathcal{Z}^d} \hat{\varphi}_c^2(\|\xi + 2\pi l\|) \Big/ \Big(\sum_{k \in \mathcal{Z}^d} \hat{\varphi}_c(\|\xi + 2\pi k\|) \Big)^2 \\ &\le 1, \end{aligned}$$

which, together with (3.5), yields inequality (3.4). ■

Thus we may define

$$\widehat{S_c f}(\xi) = \hat{\chi}_c(\xi) \sum_{k \in \mathcal{Z}^d} f(k) \exp(-ik\xi), \tag{3.7}$$

and the series is convergent in $L^2(\mathcal{R}^d)$. Applying the inverse Fourier transform term by term, we obtain the useful equation

$$S_c f(x) = \sum_{k \in \mathcal{Z}^d} f(k) \chi_c(x - k), \quad x \in \mathcal{R}^d.$$

Theorem 3.4. *Let $f \in E_\pi(\mathcal{R}^d) \cap L^2(\mathcal{R}^d)$ be a continuous function. We have*

$$\lim_{c \to \infty} S_c f(x) = f(x),$$

and the convergence is uniform on $\mathcal{R}^d$.

Proof. We have the equation

$$S_c f(x) - f(x) = (2\pi)^{-d} \int_{\mathcal{R}^d} \sum_{k \in \mathcal{Z}^d} \hat{f}(\xi + 2\pi k)\, (\hat{\chi}_c(\xi) - I(\xi)) \exp(ix\xi)\, d\xi.$$

Thus we deduce the bound

$$\begin{aligned} &|S_c f(x) - f(x)| \\ &\le (2\pi)^{-d} \int_{[-\pi,\pi]^d} |\hat{f}(\xi)| \sum_{k \in \mathcal{Z}^d} \Big| \hat{\chi}_c(\xi + 2\pi k) - I(\xi + 2\pi k) \Big|\, d\xi \\ &= (2\pi)^{-d} \int_{[-\pi,\pi]^d} |\hat{f}(\xi)| \left(1 - \hat{\chi}_c(\xi) + \sum_{k \in \mathcal{Z}^d \setminus \{0\}} \hat{\chi}_c(\xi + 2\pi k) \right) d\xi, \end{aligned} \tag{3.8}$$

using the fact that $\hat{\chi}_c$ is non-negative, and we observe that this upper bound is independent of x. Therefore we prove that the upper bound converges to zero as $c \to \infty$.

Applying (1.3), we obtain the relation

$$\sum_{k \in \mathcal{Z}^d \setminus \{0\}} \hat{\chi}_c(\xi + 2\pi k) = 1 - \hat{\chi}_c(\xi), \tag{3.9}$$

whence

$$|S_c f(x) - f(x)| \le 2(2\pi)^{-d} \int_{[-\pi,\pi]^d} |\hat{f}(\xi)|(1 - \hat{\chi}_c(\xi))\, d\xi. \tag{3.10}$$

Now $\hat{f} \in L^2([-\pi,\pi]^d)$ implies $\hat{f} \in L^1([-\pi,\pi]^d)$, by the Cauchy-Schwartz inequality. Further, Proposition 2.2 gives the limit $\lim_{c\to\infty} \hat{\chi}_c(\xi) = 1$, for $\xi \in (-\pi,\pi)^d$, and we have $0 \le 1 - \hat{\chi}_c(\xi) \le 1$, by (1.3). Therefore the dominated convergence theorem implies that

$$\lim_{c\to\infty} (2\pi)^{-d} \int_{[-\pi,\pi]^d} |\hat{f}(\xi)|(1 - \hat{\chi}_c(\xi))\, d\xi = 0.$$

The proof is complete. ■

Conclusions

Section 4 of Powell (1991) provides an explicit calculation that is analogous to the proof of Theorem 3.4 when $f(x) = x^2$. Of course, this function does not satisfy the conditions of Theorem 3.4. Therefore extensions of this result are necessary, but the final form of the theorem is not clear at present.

Theorem 3.4 encourages the use of large c for certain functions. Indeed, it suggests that large c will provide high accuracy interpolants for univariate functions that are well approximated by integer translates of the sinc function. Thus, in exact arithmetic, a large value of c should be useful whenever the function is well approximated by the Whittaker cardinal series. However, we remark that the linear systems arising when c is large can be rather ill-conditioned. Indeed, Baxter (1991) proves that the smallest eigenvalue of the interpolation matrix generated by a finite regular grid converges to zero exponentially quickly as $c \to \infty$. We refer the reader to Table I of Baxter (1991) for further information. Therefore special techniques are required for the effective use of large c.

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