Regarding the p-norms of radial basis interpolation matrices

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A radial basis function approximation has the form

$$s(x) = \sum_{j=1}^{n} y_j \, \varphi(x - x_j), \qquad x \in \mathcal{R}^d,$$

where $\varphi: \mathcal{R}^d \to \mathcal{R}$ is some given (usually radially symmetric) function, $(y_j)_1^n$ are real coefficients, and the centres $(x_j)_1^n$ are points in $\mathcal{R}^d$. For a wide class of functions φ, it is known that the interpolation matrix $A = (\varphi(x_j - x_k))_{j,k=1}^n$ is invertible. Further, several recent papers have provided upper bounds on $\|A^{-1}\|_2$, where the points $(x_j)_1^n$ satisfy the condition $\|x_j - x_k\|_2 \geq \delta$, $j \neq k$, for some positive constant δ. In this paper, we calculate similar upper bounds on $\|A^{-1}\|_p$ for $p \geq 1$ which apply when φ decays sufficiently quickly and A is symmetric and positive definite. We include an application of this analysis to a preconditioning of the interpolation matrix $A_n = (\varphi(j-k))_{j,k=1}^n$ when $\varphi(x) = (x^2 + c^2)^{1/2}$, the Hardy multiquadric. In particular, we show that $\sup_n \|A_n^{-1}\|_\infty$ is finite. Furthermore, we find that the bi-infinite symmetric Toeplitz matrix $E = (\varphi(j-k))_{j,k \in \mathcal{Z}^d}$ enjoys the remarkable property that $\|E^{-1}\|_p = \|E^{-1}\|_2$ for every $p \geq 1$ when φ is a Gaussian. Indeed, we also show that this property persists for any function φ which is a tensor product of even, absolutely integrable Pólya frequency functions.

1. Introduction

A radial basis function approximation has the form

$$s(x) = \sum_{j=1}^{n} y_j \, \varphi(x - x_j), \qquad x \in \mathcal{R}^d,$$

where $\varphi: \mathcal{R}^d \to \mathcal{R}$ is some given (usually radially symmetric) function, $(y_j)_1^n$ are real coefficients, and the centres $(x_j)_1^n$ are points in $\mathcal{R}^d$. Such approximants provide a flexible and useful approach to multivariate interpolation (see [D, P]). For a wide class of functions φ, it is known that the interpolation matrix

$$A = (\varphi(x_j - x_k))_{j,k=1}^n \tag{1.1}$$

is invertible (see [M, MN]). Further, several recent papers [B, NW1, NW2, Ba1, Ba2] have provided upper bounds on $\|A^{-1}\|_2$, where the points $(x_j)_1^n$ satisfy the separation condition $\|x_j - x_k\|_2 \geq \delta$, $j \neq k$, for some positive constant δ. In this paper we derive upper bounds on $\|A^{-1}\|_p$ for certain functions φ and $p \geq 1$, under the same separation condition on the set of centres. Specifically, in Section 2, we use total positivity to show that the bi-infinite symmetric Toeplitz matrix $E = (\exp(-\lambda \|j - k\|_2^2))_{j,k \in \mathcal{Z}^d}$ enjoys the remarkable property that $\|E^{-1}\|_p = \|E^{-1}\|_2$ for every $p \geq 1$. (We refer the reader to [GS] or [W] for the general theory of Toeplitz matrices.) Furthermore, we show that this result generalizes to a certain larger class of bi-infinite symmetric Toeplitz matrices generated by Pólya frequency functions. In Section 3, we show that a theorem of Demko, Moss and Smith on the inverse of a banded symmetric positive definite matrix can be used to provide bounds on $\|A^{-1}\|_\infty$, where A is given by (1.1), φ is a strictly positive definite function (see Definition 1.1) satisfying certain growth restrictions, and $(x_j)_1^n$ can be any set of different points in $\mathcal{R}^d$. Moreover, the symmetry of A implies $\|A^{-1}\|_1 = \|A^{-1}\|_\infty$, whence the Riesz convexity theorem [HLP, p. 214], which states that $\log \|A^{-1}\|_p$ is a convex function of $1/p$, provides an upper bound on $\|A^{-1}\|_p$ for all $p \geq 1$. The results of Section 3 do not apply directly to the many unbounded radial basis functions studied in the literature. However, we use a preconditioning argument and the main theorem of Section 3 to prove that the inverses of the matrices

$$A_n = (\varphi(j - k))_{j,k=0}^n, \qquad n = 0, 1, \ldots,$$

where $\varphi(x) = (x^2 + c^2)^{1/2}$ and c is a positive constant, are uniformly bounded in the ∞-norm. Once again, symmetry and the Riesz convexity theorem imply that the matrices $(A_n^{-1})_{n=0}^\infty$ are also uniformly bounded in the p-norm for every $p \geq 1$.

To the writers' knowledge, this is the first paper dedicated to the study of upper bounds for $\|A^{-1}\|_p$ for $p \neq 2$. However, the result for the multiquadric in Section 4 was discovered independently by M. J. D. Powell (private communication), whose technique was quite different; his study of $\|A_n^{-1}\|_\infty$ was motivated by a desire to understand the uniform error of interpolation by a multiquadric on a regular grid [BP].

We shall need the following definition and proposition in Sections 3 and 4.

Definition 1.1. We shall say that a function $f\colon \mathcal{R}^d \to \mathcal{C}$ is *positive definite* if it is measurable and

$$\sum_{j,k=1}^{n} a_j \overline{a_k} f(x_j - x_k) \geq 0 \tag{1.2}$$

for any complex sequence $(a_j)_{j=1}^n$ and any points $(x_j)_{j=1}^n$ in $\mathcal{R}^d$. Further, we shall say that f is *strictly positive definite* if inequality (1.2) is strict whenever $(a_j)_{j=1}^n$ is a non-zero sequence and the points $(x_j)_{j=1}^n$ are all different.

Proposition 1.2. *Let $f\colon \mathcal{R}^d \to \mathcal{R}$ be positive definite. Then $f(x) = f(-x)$ for every $x \in \mathcal{R}^d$.*

Proof. See [R, p.18]. ∎

2. Cardinal interpolation and norm estimates

Let λ be a positive constant and let $\varphi\colon \mathcal{R}^d \to \mathcal{R}$ be the Gaussian

$$\varphi(x) = \exp(-\lambda \|x\|_2^2), \qquad x \in \mathcal{R}^d. \tag{2.1}$$

In this section we prove the remarkable result that the bi-infinite symmetric Toeplitz matrix $A = (\varphi(j-k))_{j,k \in \mathcal{Z}^d}$ satisfies the equation

$$\|A^{-1}\|_p = \|A^{-1}\|_1$$

for every p greater than one. Using some recent work of Baxter and Micchelli, we can view this as a consequence of the fact that the Gaussian is an even, absolutely integrable Pólya frequency function, and we address this point briefly at the end of the section. Indeed, it would be possible to begin our study with this more general work, but we have chosen to concentrate on the particular case of the Gaussian because of its greater familiarity to many readers.

We begin with the case $d = 1$. Consider the bi-infinite, symmetric Toeplitz matrix $A = (\varphi(j-k))_{j,k \in \mathcal{Z}}$. Then the theory of Toeplitz matrices [W, Section 2] implies that A is an invertible bounded linear operator on $\ell^p(\mathcal{Z})$ for every $p \geq 1$ because the symbol function

$$\sigma(\xi) = \sum_{k \in \mathcal{Z}} \hat{\varphi}(\xi + 2\pi k), \qquad \xi \in \mathcal{R},$$

is a positive continuous function. Furthermore, A^{-1} is a Toeplitz matrix, say, $A^{-1} = (c_{j-k})_{j,k\in\mathcal{Z}}$. Then the *cardinal function* $\chi : \mathcal{R} \to \mathcal{R}$ given by

$$\chi(x) = \sum_{k\in\mathcal{Z}} c_k \varphi(x-k), \qquad x \in \mathcal{R}, \tag{2.2}$$

satisfies $\chi(j) = \delta_{0j}$, $j \in \mathcal{Z}$, and the exponential decay of φ ensures the absolute convergence of the series in equation (2.2) at every point x in $\mathcal{R}$.

We shall prove the following.

Theorem 2.1. $\|A^{-1}\|_p = \|A^{-1}\|_2$ *for* $p \geq 1$.

We first need a preliminary result.

Proposition 2.2. *The coefficients* $(c_k)_{k\in\mathcal{Z}}$ *of the cardinal function* χ *alternate in sign, that is* $(-1)^k c_k \geq 0$ *for every integer* k.

Proof. For each non-negative integer n, we let

$$A_n = (\varphi(j-k))_{j,k=-n}^{n}. \tag{2.3}$$

Now A_n is an invertible totally positive matrix (see [K, Section 7.1, p. 334]), that is every minor is non-negative. Therefore A_n^{-1} enjoys the "chequerboard" property: the elements of the inverse matrix satisfy $(-1)^{j+k}(A_n^{-1})_{jk} \geq 0$, for $j,k = -n,\ldots,n$. In particular, if we let

$$c_{nk} = (A_n^{-1})_{0k}, \qquad k = -n,\ldots,n, \tag{2.4}$$

then $(-1)^k c_{nk} \geq 0$ and the definition of A_n^{-1} provides the equations

$$\sum_{k=-n}^{n} c_{nk}\varphi(j-k) = \delta_{0j}, \qquad j = -n,\ldots,n. \tag{2.5}$$

Thus the function $\chi_n : \mathcal{R} \to \mathcal{R}$ defined by

$$\chi_n(x) = \sum_{k=-n}^{n} c_{nk}\varphi(x-k), \qquad x \in \mathcal{R}, \tag{2.6}$$

is a cardinal function of interpolation for the finite set $\{-n,\ldots,n\}$.

Now Theorem 9 of [BM] supplies the following useful fact relating the coefficients of χ_n and χ:

$$\lim_{n\to\infty} c_{nk} = c_k, \qquad k \in \mathcal{Z}.$$

Hence the property $(-1)^k c_{nk} \geq 0$ implies the required condition $(-1)^k c_k \geq 0$. ■

Proof of Theorem 2.1. By [W, p. 186],

$$c_k = (2\pi)^{-1} \int_0^{2\pi} \frac{1}{\sigma(\xi)}\, e^{-ik\xi}\, d\xi, \qquad k \in \mathcal{Z}, \tag{2.7}$$

where

$$\sigma(\xi) = \sum_{k \in \mathcal{Z}} \hat{\varphi}(\xi + 2\pi k), \qquad \xi \in \mathcal{R}. \tag{2.8}$$

Therefore, using standard results of Toeplitz operator theory ([W, Theorem 1']), we obtain the expression

$$\|A^{-1}\|_2 = \max\{\frac{1}{\sigma(\xi)} : \xi \in [0, 2\pi]\}.$$

Applying Lemma 2.7 of [Ba2], we get

$$\|A^{-1}\|_2 = \frac{1}{\sigma(\pi)} = \sum_{k \in \mathcal{Z}} (-1)^k c_k. \tag{2.9}$$

But Proposition 2.1 and the symmetry of A provide the relations

$$\|A^{-1}\|_\infty = \|A^{-1}\|_1 = \sum_{k \in \mathcal{Z}} |c_k| = \sum_{k \in \mathcal{Z}} (-1)^k c_k. \tag{2.10}$$

Therefore A^{-1} is a nontrivial linear operator on $\ell^p(\mathcal{Z})$, for $p = 1, 2$, and ∞, whose norms agree on each of these sequence spaces. Consequently, the Riesz convexity theorem [HLP, p. 214] implies $\|A^{-1}\|_p = \|A^{-1}\|_2$ for $p \geq 1$. ■

Turning now to the multivariate case, we let $\varphi(x) = \exp(-\lambda \|x\|_2^2)$, $\lambda > 0$, $x \in \mathcal{R}^d$, and define $A := (\varphi(j-k))_{j,k \in \mathcal{Z}^d}$. Then A may be viewed as a bounded linear operator on $\ell^p(\mathcal{Z}^d)$ for $p \geq 1$. As before, $A^{-1} = (c^{(d)}_{j-k})_{j,k \in \mathcal{Z}^d}$, where $(c^{(d)}_j)_{j \in \mathcal{Z}^d}$ are the coefficients of the multivariate cardinal function, to wit: $\sum_{k \in \mathcal{Z}^d} c^{(d)}_k \, \varphi(j-k) = \delta_{0,j}$, $j \in \mathcal{Z}^d$. The following is an extension of Theorem 2.1.

Proposition 2.3. *Let $A = (\varphi(j-k))_{j,k \in \mathcal{Z}^d}$. Then $\|A^{-1}\|_p = \|A^{-1}\|_2$ for $p \geq 1$.*

Proof. The Toeplitz theory and [Ba2, Lemma 2.8] give the relations

$$c^{(d)}_k = (2\pi)^{-d} \int_{[0,2\pi]^d} \frac{1}{\sigma^{(d)}(\xi)}\, e^{-ik\xi}\, d\xi, \qquad k = (k_1, \ldots, k_d) \in \mathcal{Z}^d \tag{2.11}$$

and

$$\|A^{-1}\|_2 = \frac{1}{\sigma(\pi, \pi, \ldots, \pi)} = \sum_{k \in \mathcal{Z}^d} (-1)^{k_1 + \cdots k_d} c^{(d)}_k,$$

where

$$\sigma^{(d)}(\xi) = \sum_{k\in\mathcal{Z}^d} \hat{\varphi}(\xi + 2\pi k). \tag{2.12}$$

Moreover,

$$\|A^{-1}\|_\infty = \|A^{-1}\|_1 = \sum_{k\in\mathcal{Z}^d} |c_k^{(d)}|. \tag{2.13}$$

Therefore it suffices to demonstrate that $(-1)^{k_1+\cdots+k_d} c_k^{(d)} \geq 0$ for all $k \in \mathcal{Z}^d$. To this end, we note that, since $\hat{\varphi}$ is a tensor product of univariate Gaussians, we have

$$\sigma^{(d)}(\xi) = \prod_{j=1}^{d} \sigma(\xi_j), \qquad \xi = (\xi_1, \ldots, \xi_d) \in \mathcal{R}^d, \tag{2.14}$$

where σ is the univariate symbol given by (2.8). Consequently, (2.11) implies

$$c_k^{(d)} = \prod_{j=1}^{d} c_{k_j}, \qquad k \in \mathcal{Z}^d, \tag{2.15}$$

whence

$$(-1)^{k_1+\cdots+k_d}\, c_k^{(d)} = \prod_{j=1}^{d} (-1)^{k_j}\, c_{k_j} \geq 0,$$

by Proposition 2.2. Once again, the Riesz convexity theorem ensures the validity of the assertion for all $p \geq 1$. ∎

Finally, we discuss the more general formulation of the results using the work of Baxter and Micchelli [BaM]. Specifically, we show that all of the results of this section apply when the Gaussian is replaced by any tensor product of even, absolutely integrable Pólya frequency functions. We recall that a function $\varphi: \mathcal{R} \to \mathcal{R}$ is a Pólya frequency function if the matrix $(\varphi(x_j - y_k))_{j,k=1}^n$ is totally positive for any real numbers $x_1 < \cdots < x_n$ and $y_1 < \cdots < y_n$.

Proposition 2.4. *Let $\varphi: \mathcal{R} \to \mathcal{R}$ be an even, absolutely integrable Pólya frequency function. Then there is an even real sequence $(c_j)_{j\in\mathcal{Z}}$ such that $\sum_{k\in\mathcal{Z}} c_k^2 < \infty$ and the function $\chi: \mathcal{R} \to \mathcal{R}$ defined by*

$$\chi(x) = \sum_{k\in\mathcal{Z}} c_k \varphi(x - k), \qquad x \in \mathcal{R}, \tag{2.16}$$

satisfies

$$\chi(j) = \delta_{0j}, \qquad j \in \mathcal{Z}.$$

Furthermore, the coefficients $(c_k)_{k\in\mathcal{Z}}$ of χ alternate in sign, that is $(-1)^k c_k \geq 0$.

Proof. In [S2] it is shown that an even, absolutely integrable Pólya frequency function must decay exponentially for large argument, which implies that the infinite series (2.16) is well defined. Standard Toeplitz theory provides the remaining properties of χ except for the alternation of signs of its coefficients, which fact requires only minor modifications to the proof of Proposition 2.2. ∎

The analogue of Theorem 2.1 is equally straightforward.

Theorem 2.5. *Let $\varphi\colon \mathcal{R} \to \mathcal{R}$ be an even, absolutely integrable Pólya frequency function and let $A = (\varphi(j-k))_{j,k\in\mathcal{Z}}$. Then $\|A^{-1}\|_p = \|A^{-1}\|_1$ for every p greater than one.*

Proof. The proof of Theorem 2.1 requires one important change to effect this result. Specifically, equation (2.9) is now a consequence of Theorem 4.2 of [BaM] instead of Lemma 2.7 of [Ba2]. ∎

For the multivariate form of this theorem, it is appropriate to define $\varphi\colon \mathcal{R}^d \to \mathcal{R}$ by the tensor product

$$\varphi(x) = \prod_{j=1}^{d} \varphi_j(x_j), \qquad x = (x_1, \ldots, x_d) \in \mathcal{R}^d, \tag{2.17}$$

where each φ_j is an even, absolutely integrable Pólya frequency function; we note that the multivariate Gaussian arises in this way from the univariate Gaussian. It is not difficult to construct functions of the form (2.17). For example, $\varphi(x) = \exp(-\|x\|_1)$ will do. We refer the reader to [BaM] for details.

As before, we consider the Toeplitz matrix

$$A = (\varphi(j-k))_{j,k\in\mathcal{Z}^d}, \tag{2.18}$$

where φ is given by (2.17). The symbol function satisfies the equation

$$\sigma(\xi) = \prod_{j=1}^{d} \sigma_j(\xi_j), \qquad \xi = (\xi_1, \ldots, \xi_d) \in \mathcal{R}^d, \tag{2.19}$$

where

$$\sigma_j(\eta) = \sum_{k\in\mathcal{Z}} \hat{\varphi}_j(\eta + 2\pi k), \qquad \eta \in \mathcal{R}. \tag{2.20}$$

Therefore the inverse of the interpolation matrix (2.18) is given by $A^{-1} = (c^{(d)}_{j-k})_{j,k\in\mathcal{Z}^d}$, where

$$\sum_{k\in\mathcal{Z}^d} c^{(d)}_k \varphi(j-k) = \delta_{0j}, \qquad j \in \mathcal{Z}^d.$$

Furthermore, (2.19) implies

$$c^{(d)}_k = \prod_{j=1}^{d} c_{jk_j}, \qquad k = (k_1, \ldots, k_d) \in \mathcal{Z}^d, \tag{2.21}$$

and $\sum_{l\in\mathcal{Z}} c_{jl}\varphi_j(k-l) = \delta_{0k}$, for $k \in \mathcal{Z}$ and $j \in \{1, 2, \ldots, d\}$. By Proposition 2.4, we have

$$(-1)^{k_1+\cdots+k_d} c_k^{(d)} = \prod_{j=1}^{d} (-1)^{k_j} c_{jk_j} \geq 0. \tag{2.22}$$

Therefore, as for the Gaussian, we have $\|A^{-1}\|_2 = \|A^{-1}\|_1$. Applying the Riesz convexity theorem yields the analogue of Proposition 2.3, which we state formally below.

Theorem 2.6. *Let $\{\varphi_j: \mathcal{R} \to \mathcal{R} : j = 1, \ldots, d\}$ be any set of even, absolutely integrable Pólya frequency functions. If $\varphi: \mathcal{R}^d \to \mathcal{R}$ is defined by the tensor product (2.17) and the bi-infinite, symmetric, Toeplitz matrix A is given by (2.18), then $\|A^{-1}\|_p = \|A^{-1}\|_1$ for every p greater than one.*

3. The main results

Let $(x_j)_{j=1}^n$ be any set of different points, or centres, in $\mathcal{R}^d$, let $\varphi: \mathcal{R}^d \to \mathcal{R}$ be an even function, and let $A = (\varphi(x_j - x_k))_{j,k=1}^n$ be the associated interpolation matrix. For many choices of φ, there exist upper bounds on $\|A^{-1}\|_2$ which depend only on φ, the dimension d of the ambient space, and the minimum separation distance $\min_{j\neq k}\|x_j - x_k\|_2$ of the centres ([NW1, NW2]). In this section we show that the same is true for every p-norm for a certain class of functions φ which includes the Gaussian. In fact, it suffices to prove this when $p = \infty$, because A is symmetric and the Riesz convexity theorem ([HLP, p. 214]) implies that $\log \|A^{-1}\|_p$ is a convex function of $1/p$ for $p \geq 1$; therefore most of this section studies $\|A^{-1}\|_\infty$.

The section falls naturally into two parts. In the first, we describe a method for obtaining upper bounds on $\|A^{-1}\|_\infty$ in terms of $\|A^{-1}\|_2$, provided the matrix A is symmetric, positive definite, and can be approximated sufficiently rapidly by banded matrices; the main theorem here is Theorem 3.6. The second part of the section deals with the application of this method to interpolation matrices, and culminates in Theorem 3.11 and Corollary 3.12.

Our main tool in the matricial analysis to follow is a theorem of Demko, Moss and Smith, given below.

Theorem 3.1. *Let $A \in \mathcal{R}^{n\times n}$ be a positive definite symmetric matrix. If A is m-banded, that is $A_{jk} = 0$ whenever $|j-k|$ exceeds m, then*

$$\left|A_{jk}^{-1}\right| \leq 2\,\|A^{-1}\|_2\,\mu^{|j-k|}, \qquad 1 \leq j, k \leq n, \tag{3.1}$$

where $\mu = \left(\frac{\sqrt{\mathrm{cond}_2(A)}-1}{\sqrt{\mathrm{cond}_2(A)}+1}\right)^{1/m}$ *and* $\mathrm{cond}_2(A) = \|A\|_2\,\|A^{-1}\|_2$.

Proof. This is Theorem 2.4 of [DMS]. ∎

We shall extend the preceding theorem to matrices indexed by d-dimensional integers. A precise definition of such matrices follows.

Definition 3.2. We shall say that a *multivariate matrix* A is simply a finitely supported function $A\colon \mathcal{Z}^d \times \mathcal{Z}^d \to \mathcal{R}$. Writing A_{jk} for $A(j,k)$, the following terms extend in a natural way.

(i) A is *symmetric* if $A_{jk} = A_{kj}$ for all $j,k \in \mathcal{Z}^d$.

(ii) A is *positive definite* if

$$\sum_{j,k\in\mathcal{Z}^d} y_j y_k A_{jk} > 0 \tag{3.2}$$

for every finitely supported non-zero real sequence $(y_j)_{j\in\mathcal{Z}^d}$ such that the support of the matrix $(y_j y_k)_{j,k\in\mathcal{Z}^d}$ is contained in the support of A.

(iii) A is R-banded, where R is a positive constant, if $A_{jk} = 0$ whenever $\|j-k\|_2$ exceeds R. (This notion of bandedness for multivariate matrices is highly useful in other contexts. See, for instance, [de B, p. 43].)

Armed with these concepts, it is possible to extend Theorem 3.1 to multivariate matrices.

Theorem 3.3. *Let $A = (A_{jk})_{j,k\in\mathcal{Z}^d}$ be a symmetric positive definite matrix which is m-banded, where m is a positive integer. Then*

$$|A^{-1}_{jk}| \le 2\|A^{-1}\|_2 \mu^{\|j-k\|_2}, \qquad j,k \in \mathcal{Z}^d, \tag{3.3}$$

where $\mu = \left(\frac{\sqrt{\mathrm{cond}_2(A)}-1}{\sqrt{\mathrm{cond}_2(A)}+1}\right)^{1/m}$.

Proof. The proof of Theorem 2.4 of [DMS] requires only minor changes to effect this result. Specifically, we note that if A is m-banded then A^l is lm-banded for any positive integer l. ∎

Theorem 3.3 provides the following upper bound on $\|A^{-1}\|_\infty$ for symmetric, positive definite, banded multivariate matrices A.

Corollary 3.4. *Let A be as in Theorem 3.3. Then*

$$\|A^{-1}\|_\infty \le 2\|A^{-1}\|_2 \left(\frac{1+\mu^{1/\sqrt{d}}}{1-\mu^{1/\sqrt{d}}}\right)^d = 2\|A^{-1}\|_2 \left(\tanh\left[\frac{1}{2md}\log\left(\frac{\kappa+1}{\kappa-1}\right)\right]\right)^{-d}, \tag{3.4}$$

where $\kappa = \sqrt{\mathrm{cond}_2(A)}$. *Further, if* $\|A^{-1}\|_2 \le \alpha$ *and* $\mathrm{cond}_2(A) \le \beta$, *then*

$$\|A^{-1}\|_\infty \le 2\alpha\left(\frac{1+\nu}{1-\nu}\right)^d, \tag{3.5}$$

where $\nu = \left(\frac{\sqrt{\beta}-1}{\sqrt{\beta}+1}\right)^{1/m\sqrt{d}}$.

Proof. The Cauchy-Schwarz inequality and (3.3) provide the relations

$$\begin{aligned}\|A^{-1}\|_\infty \le 2\|A^{-1}\|_2 \sum_{k\in\mathcal{Z}^d} \mu^{\|k\|_2} &\le 2\|A^{-1}\|_2 \sum_{k\in\mathcal{Z}^d} \mu^{\|k\|_1/\sqrt{d}} \\ &= 2\|A^{-1}\|_2 \left(\sum_{l=-\infty}^{\infty} \mu^{|l|/\sqrt{d}}\right)^d = 2\|A^{-1}\|_2 \left(\frac{1+\mu^{1/\sqrt{d}}}{1-\mu^{1/\sqrt{d}}}\right)^d.\end{aligned}$$

Using the identity

$$\frac{1-\mu^{1/\sqrt{d}}}{1+\mu^{1/\sqrt{d}}} = \tanh\left[\frac{1}{2m\sqrt{d}}\log\left(\frac{\kappa+1}{\kappa-1}\right)\right]$$

provides the remainder of (3.4). Furthermore, it is elementary that the function

$$[0,\infty) \ni x \mapsto \left(\tanh\left[\frac{1}{2m\sqrt{d}}\log\left(\frac{x+1}{x-1}\right)\right]\right)^{-d}$$

is increasing, which implies the second part of this corollary. ∎

We wish to use Corollary 3.4 to obtain upper bounds on $\|A^{-1}\|_\infty$ when A is a symmetric, positive definite multivariate matrix which can be approximated sufficiently rapidly by banded matrices. We need the following standard perturbation result.

Lemma 3.5. *Let $(V, \|\cdot\|)$ be any normed space and let $S: V \to V$ be an invertible bounded linear operator whose inverse is also bounded. If $T: V \to V$ is a linear operator such that $\|S - T\| \le 1/(2\|S^{-1}\|)$, then $\|T^{-1}\| \le 2\|S^{-1}\|$.*

Proof. We have the relations

$$\|Tx\| \ge \|Sx\| - \|(S-T)x\| \ge \|x\|/\|S^{-1}\| - \|x\|/(2\|S^{-1}\|) = \|x\|/(2\|S^{-1}\|).$$

∎

We now address the main result of the first part of this section.

Theorem 3.6. *Suppose $A = (A_{jk})_{j,k\in\mathcal{Z}^d}$ is a symmetric, positive definite matrix. Let $(A_m)_{m=1}^\infty$, be a sequence of symmetric multivariate matrices such that*

(i) A_m is m-banded for each positive integer m;

(ii) $\sup_m \|A_m\|_\infty =: K < \infty$;

(iii) $\|A - A_m\|_\infty = O(m^{-t+d})$, for some constant $t > 2d$.

Then there exists an increasing function $D : [0,\infty) \to [0,\infty)$ such that $\|A^{-1}\|_\infty \le D(\|A^{-1}\|_2)$.

Proof. Suppose $\|x\|_2 = 1$. The Cauchy-Schwarz inequality and the symmetry of $A - A_m$ yield the pair of inequalities

$$|x^T Ax - x^T A_m x| \le \|A - A_m\|_2 \le \|A - A_m\|_\infty.$$

So, by assumption (iii) and positive definiteness of A, there exists a positive integer m_0 such that, for every $m \ge m_0$, A_m is positive definite and $\|A - A_m\|_2 \le 1/(2\|A^{-1}\|_2)$. Consequently, Lemma 3.5 implies that $\|A_m^{-1}\|_2 \le 2\|A^{-1}\|_2$ for $m \ge m_0$, so from assumption (ii), we have the estimate $\mathrm{cond}_2(A_m) \le 2K\|A^{-1}\|_2 =: \beta$, $m \ge m_0$. Hence, we conclude from Corollary 3.4 that

$$\|A_m^{-1}\|_\infty \le 4\|A^{-1}\|_2 \left(\frac{1+\nu}{1-\nu}\right)^d, \quad m \ge m_0, \tag{3.6}$$

where

$$\nu = \left(\frac{\sqrt{\beta}-1}{\sqrt{\beta}+1}\right)^{1/(m\sqrt{d})}.$$

Applying (3.4), we find that the right hand side of (3.6) is $O(m^d)$, so, since $t > 2d$, assumption (iii) guarantees the existence of a positive integer $m_1 \ge m_0$ such that $\|A - A_m\|_\infty \le 1/(2\|A_m^{-1}\|_\infty)$ for all $m \ge m_1$. An appeal to Lemma 3.5 and inequality (3.6) now provides the following final estimate:

$$\|A^{-1}\|_\infty \le 2\|A_m^{-1}\|_\infty \le 8\|A^{-1}\|_2 \left(\frac{1+\nu}{1-\nu}\right)^d, \quad m \ge m_1, \tag{3.7}$$

where

$$\nu = \left(\frac{\sqrt{\beta}-1}{\sqrt{\beta}+1}\right)^{1/(m\sqrt{d})}.$$

The observation that the right hand side of (3.7) is an increasing function of $\|A^{-1}\|_2$ completes the proof. ∎

We now apply the results obtained heretofore to interpolation matrices of the form $(\varphi(x_j - x_k))_{j,k=1}^n$, where $(x_j)_{j=1}^n$ is any set of different points in $\mathcal{R}^d$ and $\varphi\colon \mathcal{R}^d \to \mathcal{R}$ is an even function. In order to do so, let us re-express the interpolation matrix as a multivariate matrix of the form $A = (A_{jk})_{j,k\in\mathcal{Z}^d}$. Roughly speaking, we choose a scaling $z_j = \rho x_j$, for $j = 1, \ldots, n$, and label each z_j with a nearby member of the lattice $\mathcal{Z}^d$. More precisely, we use the following elegant lemma of D. Hensley.

Lemma 3.7. *Let $(z_j)_{j=1}^n$ be any subset of $\mathcal{R}^d$ such that $\|z_j - z_k\|_2 \ge \sqrt{d}$ when $j \ne k$. Define*

$$\nu_j = \left(\lfloor \xi_1^j \rfloor, \ldots, \lfloor \xi_d^j \rfloor\right), \qquad j = 1, \ldots, n,$$

where $z_j = (\xi_1^j, \ldots, \xi_d^j)$ and $\lfloor\cdot\rfloor$ denotes the greatest integer function. Then the points $(\nu_j)_{j=1}^n$ are all different. Further, if $\|\nu_j - \nu_k\|_2 \ge R$ and $R \ge 4\sqrt{d}$, then $\|z_j - z_k\|_2 \ge R/2$.

Proof. If $\nu_j = \nu_k$, then $|\xi_l^j - \xi_l^k| < 1$ for $l = 1, \ldots, d$, which implies $\|z_j - z_k\|_2 < \sqrt{d}$. Therefore $j = k$ and we have shown that the integers $(\nu_j)_{j=1}^n$ are all different. For the second statement of the lemma, we note that

$$\|\nu_j - \nu_k\|_2 \le \|z_j - z_k\|_2 + \|(z_j - \nu_j) - (z_k - \nu_k)\|_2 < \|z_j - z_k\|_2 + 2\sqrt{d},$$

whence $\|\nu_j - \nu_k\|_2 \ge R$ implies $\|z_j - z_k\|_2 \ge R - 2\sqrt{d} \ge R/2$ for $R \ge 4\sqrt{d}$. ∎

In order to apply Lemma 3.7 to arbitrary sets of distinct points, we introduce the following terminology.

Definition 3.8. Let q be a positive number. A subset X of $\mathcal{R}^d$ is said to be q-separated if the open balls $\{B_x\}_{x \in X}$ are disjoint, where $B_x = \{y \in \mathcal{R}^d : \|x - y\|_2 < q\}$.

We are now ready to define the multivariate matrix corresponding to $(\varphi(x_j - x_k))_{j,k=1}^n$. Given any q-separated subset $(x_j)_{j=1}^n$ in $\mathcal{R}^d$, we set $z_j = (\sqrt{d}/2q)x_j$ for $j = 1, \ldots, n$, and calculate integers $(\nu_j)_{j=1}^n$ using Lemma 3.7. Define $A = (A_{jk})_{j,k \in \mathcal{Z}^d}$ by the equations

$$A_{jk} = \begin{cases} \varphi(x_l - x_m), & \text{if } j = \nu_l \text{ and } k = \nu_m \\ 0, & \text{otherwise} \end{cases}. \tag{3.8}$$

Thus, if φ is a strictly positive definite function of compact support, then (3.8) produces a banded, positive definite, symmetric multivariate matrix A. Unfortunately, whilst there are several strictly positive definite radial basis functions in common use, such as the Gaussian $\varphi(x) = \exp(-\lambda\|x\|_2^2)$ and the inverse multiquadric $\varphi(x) = (\|x\|_2^2 + c^2)^{1/2}$, not one enjoys the property of compact support. However, if φ decays rather quickly for large argument, then it can be useful to approximate φ by the compactly supported function given by

$$\varphi_m(x) := \begin{cases} \varphi(x), & \|x\|_2 \le m \\ 0 & \|x\|_2 > m \end{cases},$$

where m is some positive integer. The function φ_m gives rise to the interpolation matrix

$$A_m := (\varphi_m(x_j - x_k))_{j,k=1}^n, \tag{3.9}$$

which can also be written as a multivariate matrix using φ_m in place of φ in (3.8). Moreover, A_m is m-banded, and we shall use it to approximate the original matrix A.

The analysis of the banded sections A_m requires the next two lemmata.

Lemma 3.9. *Let X be any q-separated subset of $\mathcal{R}^d$ contained in the annulus $\{x \in \mathcal{R}^d : r_1 \le \|x\|_2 \le r_2\}$, where $q \le r_1 \le r_2$. Let $\varphi\colon \mathcal{R}^d \to \mathcal{R}$ be a function such that*

$$0 \le \varphi(x) \le C\|x\|_2^{-t}, \qquad x \in \mathcal{R}^d,$$

where C and $t > d$ are positive constants. Then

$$\sum_{x\in X}\varphi(x) \le 3dq^{-t}C \sum_{l=\lfloor r_1/q\rfloor}^{\lfloor r_2/q\rfloor} l^{-t}(l+2)^{d-1} \le 3dq^{-t}C\sum_{l=1}^{\infty} l^{-t}(l+2)^{d-1} < \infty. \tag{3.10}$$

Proof. We have

$$\sum_{x\in X}\varphi(x) \le C\sum_{x\in X}\|x\|_2^{-t} \le q^{-t}C \sum_{l=\lfloor r_1/q\rfloor}^{\lfloor r_2/q\rfloor} l^{-t}\text{card}\{x\in X : ql \le \|x\|_2 \le q(l+1)\}.$$

Now $\text{card}\{x \in X : ql \le \|x\|_2 \le q(l+1)\}$ cannot be larger than the maximum number of disjoint open balls of radius q which can be packed in the annulus $\{x \in \mathcal{R}^d : q(l-1) \le \|x\|_2 \le q(l+2)\}$, and comparing the volumes of this latter annulus with that of an open ball of radius q we obtain the upper bound $(l+2)^d - (l-1)^d \le 3d(l+2)^{d-1}$. Hence

$$\sum_{x\in X}\varphi(x) \le 3dq^{-t}C \sum_{l=\lfloor r_1/q\rfloor}^{\lfloor r_2/q\rfloor} l^{-t}(l+2)^{d-1} \le 3dq^{-t}C\sum_{l=1}^{\infty} l^{-t}(l+2)^{d-1} < \infty,$$

the condition $t > d$ implying the finiteness of this final bound. ∎

Lemma 3.10. *Let $(x_j)_{j=1}^n$ be any q-separated subset of $\mathcal{R}^d$ and let $A = (\varphi(x_j - x_k))_{j,k=1}^n$ where $\varphi\colon \mathcal{R}^d \to \mathcal{R}$ is as in Lemma 3.9. Suppose A_m is the banded matrix given by (3.9) for every positive integer m. Then $\sup_m \|A_m\|_\infty < \infty$ and $\|A - A_m\|_\infty = O(m^{-t+d})$.*

Proof. Standard linear algebra and the definition of φ_m imply the relations

$$\|A_m\|_\infty \le \|A\|_\infty = \varphi(0) + \max_{1\le j\le n} \sum_{k=1,k\ne j}^{n} \varphi(x_j - x_k). \tag{3.11}$$

It is easy to check that the points $\{x_j - x_k\}_{k=1,k\ne j}^n$ form a q-separated set of $\mathcal{R}^d$ satisfying the hypotheses of Lemma 3.9. Consequently, (3.10) and (3.11) provide the inequality

$$\|A_m\|_\infty \le \varphi(0) + 3dq^{-t}C\sum_{l=1}^{\infty} l^{-t}(l+2)^{d-1} < \infty \tag{3.12}$$

for every m. A similar argument leads to the estimate

$$\|A - A_m\|_\infty \le 3dq^{-t}C \sum_{l=\lfloor m/q\rfloor}^{\infty} l^{-t}(l+2)^{d-1}, \tag{3.13}$$

whence the asserted rate of convergence. ∎

Note that (3.12) and (3.13) do not depend on the particular choice of q-separated set $(x_j)_{j=1}^n$.

We close the section with our main results for interpolation matrices.

Theorem 3.11. *Let $\varphi: \mathcal{R}^d \to \mathcal{R}$ be a strictly positive definite function such that*

$$0 \le \varphi(x) \le C\|x\|_2^{-t}, \qquad x \in \mathcal{R}^d,$$

where C and $t > 2d$ are constants. Then there is an increasing function $D: [0, \infty) \to [0, \infty)$ such that, for every finite q-separated subset $(x_j)_{j=1}^n$ of $\mathcal{R}^d$, the interpolation matrix $A = (\varphi(x_j - x_k))_{j,k=1}^n$ satisfies the inequality

$$\|A^{-1}\|_\infty \le D(\|A^{-1}\|_2).$$

Proof. Let A_m be as above and regard A and A_m as multivariate matrices. The required result follows from Lemma 3.10 and Theorem 3.6. ∎

Corollary 3.12. *Let $\varphi: \mathcal{R}^d \to \mathcal{R}$ satisfy the conditions of Theorem 3.11 and let there be a constant $K(q)$ such that*

$$\|A^{-1}\|_2 \le K(q), \tag{3.14}$$

where $A = (\varphi(x_j - x_k))_{j,k=1}^n$ and $(x_j)_{j=1}^n$ can be any finite q-separated subset of $\mathcal{R}^d$. Then there is a constant $L(q)$ for which

$$\|A^{-1}\|_p \le L(q), \tag{3.15}$$

for every $p \ge 1$.

Proof. This follows immediately from Theorem 3.11 when $p = \infty$, the result for $p = 1$ being an obvious consequence of symmetry. The Riesz convexity theorem then provides the result for all p greater than one. ∎

We refer the reader to [NW1, NW2] for bounds such as (3.14).

4. An application to the Hardy multiquadric

Let $\varphi: \mathcal{R} \to \mathcal{R}$ be the Hardy multiquadric, that is

$$\varphi(x) = (x^2 + c^2)^{1/2}, \qquad x \in \mathcal{R}, \tag{4.1}$$

where c is a non-negative constant, and let $A_n = (\varphi(j - k))_{j,k=0}^n$. In this section we prove that $\sup_n \|A_n^{-1}\|_\infty$ is finite for every $c \ge 0$. This is not a simple consequence of Section 3 because the Hardy multiquadric is not a positive definite function. However, we find that it is suitable to apply the results of Section 3 to the second divided difference of the multiquadric in the rather special case of equally spaced points on a line, the general case being unclear at this time. We emphasize that the convergence analysis of interpolation at equally spaced points on a line is still a topic for current research. In particular, the papers of [BP, P] are devoted to this problem. Furthermore, [P] finds that there is an intimate link between uniform convergence of the multiquadric interpolants and the boundedness of the set $\{\|A_n^{-1}\|_\infty : n = 1, 2, \dots\}$. Therefore there are several reasons to study bounds on $\|A_n^{-1}\|_\infty$.

When $c = 0$, A_n becomes the Euclidean distance matrix $(|j-k|)_{j,k=0}^n$. Direct calculation provides the inverse matrix

$$\begin{pmatrix} (1-n)/2n & 1/2 & & & & 1/2n \\ 1/2 & -1 & 1/2 & & & \\ & 1/2 & -1 & & & \\ & & & \ddots & & \\ & & & & -1 & 1/2 \\ 1/2n & & & & 1/2 & (1-n)/2n \end{pmatrix}. \tag{4.2}$$

Thus $\|A_n^{-1}\|_\infty = 2$ for every n when $c = 0$. Therefore we restrict attention to the case when c is positive.

Our technique rests on the observation that the second divided difference

$$\psi(x) = \frac{1}{2}\left(\varphi(x+1) - 2\varphi(x) + \varphi(x-1)\right), \qquad x \in \mathcal{R}, \tag{4.3}$$

generates matrices $C_n = (\psi(j-k))_{j,k=0}^n$ which are amenable to the analysis of Section 3. Specifically we have the following pair of results.

Lemma 4.1. *The condition numbers* $(\mathrm{cond}_2(C_n))_{n=0}^\infty$ *form a bounded set.*

Lemma 4.2. *There is a constant* Γ *such that*

$$\|C_n^{-1}\|_\infty \le \Gamma, \qquad n \ge 0. \tag{4.4}$$

We shall prove these lemmata later in order to continue our main argument.

Let us introduce a new matrix B_n by the equation

$$B_n = C_n A_n^{-1}, \qquad n \ge 0, \tag{4.5}$$

recalling that A_n is invertible by [M, Theorem 2.3]. The particular form of C_n allows us to calculate B_n rather easily. Indeed we find that

$$B_n = \begin{pmatrix} & & \alpha_n^T & & & \\ 1/2 & -1 & 1/2 & & & \\ & 1/2 & -1 & & & \\ & & & \ddots & & \\ & & & & -1 & 1/2 \\ & & \beta_n^T & & & \end{pmatrix}, \tag{4.6}$$

where

$$A_n\alpha_n = \begin{pmatrix} \psi(0) \\ \psi(1) \\ \vdots \\ \psi(n) \end{pmatrix} =: \gamma_n \tag{4.7}$$

and $(\beta_n)_j = (\alpha_n)_{n-j}$ for $j = 0, 1, \ldots, n$. Thus (4.7) implies the inequalities

$$\|B_n\|_1 \le 2 + 2 \max_{0\le j\le n} |(\alpha_n)_j| \le 2 + 2\|A_n^{-1}\|_2 \cdot \|\gamma_n\|_2. \tag{4.8}$$

Now an application of the mean value theorem provides the relation $\psi(x) = \mathcal{O}(|x|^{-3})$ for large $|x|$, which allows us to conclude that

$$\|\gamma_n\|_2^2 \le \sum_{j=0}^{\infty} \psi(j)^2 = \mathcal{O}(\sum_{j=1}^{\infty} j^{-6}) < \infty. \tag{4.9}$$

Consequently $\sup_n \|\gamma_n\|_2$ is finite. Furthermore, $\sup_n \|A_n^{-1}\|_2$ is also finite (see [NW2] or [Ba2, Proposition 4.2]). Hence there is a constant Δ, depending only on c, such that

$$\|B_n\|_1 \le \Delta, \qquad n \ge 0. \tag{4.10}$$

Applying (4.5), (4.10), Lemma 4.2, and the symmetry of C_n, we obtain the relations

$$\|A_n^{-1}\|_\infty = \|A_n^{-1}\|_1 \le \|C_n^{-1}\|_1 \|B_n\|_1 \le \Gamma\Delta, \tag{4.11}$$

for every non-negative integer n.

Finally, we address the proofs of Lemmata 4.1 and 4.2. It is easy to see that $\sup_n \mathrm{cond}_2(C_n)$ is finite using the classical theory of Toeplitz operators, because the symbol function $\sigma(\xi) = \sum_{k\in\mathcal{Z}^d} \hat\psi(\xi + 2\pi k)$ is clearly a positive continuous function. However, we include a sharp bound on $\mathrm{cond}_2(C_n)$ for the enjoyment of the reader.

Lemma 4.3. *Let $\Phi: \mathcal{R}^d \to \mathcal{R}$ be the d-dimensional multiquadric*

$$\Phi(x) = \left(\|x\|_2^2 + c^2\right)^{1/2}, \qquad x \in \mathcal{R}^d.$$

Then the function $\hat\Psi: \mathcal{R}^d \to \mathcal{R}$ given by

$$\hat\Psi(\xi) = -2\sin^2(\|\xi\|_2/2)\hat\Phi(\xi), \qquad \xi \in \mathcal{R}^d, \tag{4.12}$$

is absolutely integrable. Further, its inverse Fourier transform

$$\Psi(x) = (2\pi)^{-d} \int_{\mathcal{R}^d} \hat\Psi(\xi) e^{ix\xi}\, d\xi, \qquad x \in \mathcal{R}^d, \tag{4.13}$$

is a uniformly continuous function which is radially symmetric and strictly positive definite.

Proof. The Fourier transform $\hat{\Phi}$ is known to be a non-positive radially symmetric function decaying exponentially for large argument for which the limit $\lim_{\|\xi\|_2 \to 0} \|\xi\|_2^{d+1} \hat{\Phi}(\xi)$ exists [Ba2, Section 4]. This implies that $\hat{\Psi}$ is an absolutely integrable radially symmetric function, because the existence of the limit $\lim_{\|\xi\|_2 \to 0} \|\xi\|_2^{d-1} \hat{\Psi}(\xi)$ ensures that $\hat{\Psi}$ is summable on every bounded neighbourhood of zero, and $\hat{\Psi}$ inherits the exponential decay of $\hat{\Phi}$ for large argument. Thus (4.13) defines a uniformly continuous radially symmetric function $\Psi: \mathcal{R}^d \to \mathcal{R}$. Further, Ψ is positive definite because (4.13) yields the relations

$$\sum_{j,k=1}^{n} y_j \overline{y_k} \Psi(x_j - x_k) = (2\pi)^{-d} \int_{\mathcal{R}^d} \Big| \sum_{j=1}^{n} y_j e^{i x_j \xi} \Big|^2 \hat{\Psi}(\xi)\, d\xi \geq 0,$$

for any complex numbers $(y_j)_{j=1}^n$ and for any points $(x_j)_{j=1}^n$ in $\mathcal{R}^d$. Moreover, if the sequence $(y_j)_{j=1}^n$ is non-zero and if the points $(x_j)_{j=1}^n$ are all different, then the function $\mathcal{C}^d \ni \xi \mapsto \sum_{j=1}^n y_j e^{i x_j \xi}$ is a non-zero entire function of d complex variables. Hence, its zero-set $\{\xi \in \mathcal{R}^d : \sum_{j=1}^n y_j e^{i x_j \xi} = 0\}$ has measure zero. Finally, since $\hat{\Phi}$ is negative almost everywhere, we deduce that the last inequality is strict, which implies that Ψ is strictly positive definite. ∎

This result is relevant because $\Phi = \varphi$ and $\Psi = \psi$ when $d = 1$. Thus Lemma 4.3 reveals that ψ is the univariate form of a radially symmetric function which is strictly positive definite for any dimension d. The proof of Lemma 4.1 may now be completed using [Ba2, Theorem 4.1] as follows.

Proof of Lemma 4.1. Lemma 4.3 and [Ba2, Theorem 3.10] imply that the symbol function $\sigma: \mathcal{R}^d \to \mathcal{R}$ defined by the equation

$$\sigma(\xi) = \sum_{k \in \mathcal{Z}^d} \hat{\Psi}(\xi + 2\pi k), \qquad \xi \in \mathcal{R}^d, \tag{4.14}$$

satisfies the inequalities

$$\sigma(\pi e) \leq \sigma(\xi) \leq \sigma(0), \qquad \xi \in \mathcal{R}^d, \tag{4.15}$$

where $e = [1, 1, \ldots, 1]^T \in \mathcal{R}^d$. Further, the theory of Toeplitz operators described in [Ba2, Section 1] yields the bounds

$$\|C_n\|_2 \leq \sigma(0) \qquad \text{and} \qquad \|C_n^{-1}\|_2 \leq 1/\sigma(\pi), \qquad n = 0, 1, 2, \ldots, \tag{4.16}$$

when $d = 1$. Hence $\text{cond}_2(C_n) \leq \sigma(0)/\sigma(\pi)$ for every non-negative integer n, and this is best possible. ∎

Proof of Lemma 4.2. This is a simple application of Theorem 3.11, since we have already seen that ψ is a positive definite function which decays cubically for large argument, that is $\psi(x) = \mathcal{O}(|x|^{-3})$ as $|x| \to \infty$. ∎

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