FUNCTIONALS OF EXPONENTIAL BROWNIAN MOTION AND DIVIDED DIFFERENCES

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Dedicated to Mike Powell on his 70^{th} birthday

Abstract. We provide a surprising new application of classical approximation theory to a fundamental asset-pricing model of mathematical finance. Specifically, we calculate an analytic value for the correlation coefficient between exponential Brownian motion and its time average, and we find the use of divided differences greatly elucidates formulae, providing a path to several new results. As applications, we find that this correlation coefficient is always at least $1/\sqrt{2}$ and, via the Hermite–Genocchi integral relation, demonstrate that all moments of the time average are certain divided differences of the exponential function. We also prove that these moments agree with the somewhat more complex formulae obtained by Oshanin and Yor.

1. Introduction

We begin with exponential, or geometric, Brownian motion, defined by

$$S(t) = e^{(r-\frac{\sigma^2}{2})t+\sigma B(t)}, \qquad t \geq 0, \tag{1.1}$$

where $B : [0, \infty) \to \mathbb{R}$ is Brownian motion. In other words, B is a stochastic process, or *random function*, for which $B(0) = 1$, its increments are independent, and, for $0 \leq s < t$, the increment $B(t) - B(s)$ is normally distributed with mean zero and variance $t - s$. The basic properties of Brownian motion are explained in Section 37 of Billingsley (1995), while Karatzas and Shreve (1991) is a comprehensive treatise. At a more elementary level, Norris (1998) provides a lucid derivation of the main properties of Brownian motion, whilst Higham (2004) provides a more general introduction well-suited to the numerical analyst.

We shall study the *time average*

$$A(T) := \frac{1}{T} \int_0^T S(t)\, dt \tag{1.2}$$

using the calculus of divided differences, a fundamental tool in approximation theory. We will, in particular, show that the correlation coefficient between $A(T)$ and $S(T)$, the moments of $A(T)$ and, more generally, joint moments of $S(T)$ and $A(T)$ can be elegantly, and usefully, expressed in terms of divided differences of the exponential function. Now the time average $A(T)$ has been extensively studied in the literature of Asian options; see, for instance, Yor (1992), Yor (2001) and Oshanin, Mogutov and Moreau (1993). However, we find that our use of divided differences both simplify and elucidate formulae. In Section 2, we derive the correlation coefficient for $S(T)$ and $A(T)$, finding that it is always at least $1/\sqrt{2}$, thus explaining the relative high correlation that is observational folklore in the financial community.[1] In Section 3, we demonstrate that the divided differences occurring in the lower moments of $S(T)$ and $A(T)$ generalise to all moments, using the fact

[1] I am particularly grateful to my friend and colleague Dr Dirk Siegel, a fellow PhD student of Mike Powell, who made this observation known to me when consulting for IBM.

that the integral of an exponential function over a simplex can be expressed, via the Hermite–Gennocchi formula, as a certain divided difference of the exponential function. In Section 4, we provide the divided difference theory required by the paper; some of this can, of course, be found in the dedicatee's excellent textbook Powell (1981), but derivations of the Hermite–Gennochi and Leibniz formula are less easily available. Therefore we have provided their brief derivations in the hope that this will enhance the paper's use to both the mathematical finance and the numerical analysis communities. Finally, in Section 5, we use our divided difference approach to derive a recurrence relation for the moments of $A(T)$.

We first observe the familiar result

$$\mathbb{E}S(T) = e^{(r-\sigma^2/2)T}\mathbb{E}e^{\sigma T^{1/2} Z} = e^{(r-\sigma^2/2)T} e^{\sigma^2 T/2} = e^{rT}. \tag{1.3}$$

Here Z denotes a generic $N(0,1)$ Gaussian random variable and we have used the standard fact that

$$\mathbb{E}e^{\lambda Z} = (2\pi)^{-1/2}\int_{\mathbb{R}} e^{\lambda\tau}e^{-\tau^2/2}\,d\tau = (2\pi)^{-1/2}\int_{\mathbb{R}} e^{-\frac{1}{2}\{(\tau-\lambda)^2-\lambda^2\}}\,d\tau = e^{\lambda^2/2}. \tag{1.4}$$

Similarly,

$$\begin{aligned}\mathbb{E}A(T) &= T^{-1}\int_0^T \mathbb{E}S(t)\,dt \\ &= \frac{e^{rT}-1}{rT}. \end{aligned} \tag{1.5}$$

The approximation theorist will immediately recognise the divided difference

$$\mathbb{E}A(T) = \exp[0, rT], \tag{1.6}$$

but a sceptical reader might view this as mere coincidence; in fact, it is but the tip of an iceberg. We remind the reader that $f[a_0, a_1, \ldots, a_n]$ is the highest coefficient of the unique polynomial of degree n interpolating f at distinct points $a_0, \ldots, a_n \in \mathbb{R}$, which implies $f[a_0] = f(a_0)$ and

$$f[a_0, a_1] = \frac{f(a_1) - f(a_0)}{a_1 - a_0}.$$

Further, it is evident that a divided difference does not depend on the order in which the points $a_0, a_1, \ldots, a_n$ are chosen. As mentioned above, Section 4 collects further divided difference theory required by this paper.

2. The correlation coefficient between the time average and the asset

We shall compute the correlation coefficient between $S(T)$ and $A(T)$. Specifically, we calculate

$$R := \frac{\mathbb{E}\left(S(T)A(T)\right) - \mathbb{E}\left(S(T)\right)\mathbb{E}\left(A(T)\right)}{\sqrt{\operatorname{var} S(T)\operatorname{var} A(T)}}. \tag{2.1}$$

We find an elegant divided difference expression for R.

Theorem 2.1. *The correlation coefficient* (2.1) *is given by*

$$R \equiv R(rT, \sigma^2 T) = \frac{\exp[rT, 2rT, (2r+\sigma^2)T]}{\sqrt{2\exp[2rT, (2r+\sigma^2)T]\exp[0, rT, 2rT, (2r+\sigma^2)T]}}. \tag{2.2}$$

Let us begin our derivation.

Lemma 2.2. *If $0 \le a \le b$, then*

$$\mathbb{E} S(a)S(b) = \exp\Big(a(r+\sigma^2) + br\Big). \tag{2.3}$$

Proof. We have

$$\begin{aligned}
\mathbb{E} S(a)S(b) &= \mathbb{E} S(a)^2 e^{(b-a)(r-\sigma^2/2)+\sigma(B(b)-B(a))} \\
&= \mathbb{E} S(a)^2 \mathbb{E} e^{(b-a)(r-\sigma^2/2)+\sigma\sqrt{b-a}Z} \\
&= e^{(2r+\sigma^2)a} e^{(b-a)r} \\
&= e^{a(r+\sigma^2))} e^{br}, \qquad (2.4)
\end{aligned}$$

where $Z \sim N(0,1)$ and we have used (1.3). □

Proposition 2.3. *We have*

$$\mathbb{E} S(T)A(T) = \exp[rT, (2r+\sigma^2)T]. \tag{2.5}$$

Proof. Applying Lemma 2.2, we obtain

$$\begin{aligned}
\mathbb{E} S(T)A(T) &= T^{-1} \int_0^T \mathbb{E} S(t)S(T)\, dt \\
&= T^{-1} \int_0^T e^{(r+\sigma^2)t} e^{rT}\, dt \\
&= \exp[rT, (2r+\sigma^2)T].
\end{aligned}$$

□

Proposition 2.4.

$$\mathbb{E}(A(T)^2) = 2\exp[0, rT, (2r+\sigma^2)T]. \tag{2.6}$$

Proof. We find

$$\begin{aligned}
\mathbb{E}(A(T)^2) &= T^{-2} \int_0^T \Big(\int_0^T \mathbb{E} S(t_1)S(t_2)\, dt_2 \Big)\, dt_1 \\
&= 2T^{-2} \int_0^T \Big(\int_0^{t_1} \mathbb{E} S(t_1)S(t_2)\, dt_2 \Big)\, dt_1. \qquad (2.7)
\end{aligned}$$

Thus

$$\begin{aligned}
\mathbb{E}(A(T)^2) &= 2T^{-2} \int_0^T \Big(\int_0^{t_1} e^{r(t_1+t_2)} e^{\sigma^2 t_2}\, dt_2 \Big)\, dt_1 \\
&= 2T^{-2} \int_0^T e^{rt_1} \Big(\frac{e^{(r+\sigma^2)t_1} - 1}{r+\sigma^2} \Big)\, dt_1 \\
&= \frac{2}{(r+\sigma^2)T} \big[\exp[0, (2r+\sigma^2)T] - \exp[0, rT] \big] \\
&= 2\exp[0, rT, (2r+\sigma^2)T], \qquad (2.8)
\end{aligned}$$

using the divided difference recurrence relation (4.1) to obtain the final line. □

Any reader still doubtful of the simplification provided by divided difference notation might consider the alternative expression provided in Hull (2000):

$$\mathbb{E}\left(A(T)^2\right) = \frac{2e^{(2r+\sigma^2)T}}{(r+\sigma^2)(2r+\sigma^2)T^2} + \frac{2}{rT^2}\left(\frac{1}{2r+\sigma^2} - \frac{e^{rT}}{r+\sigma^2} \right).$$

There is a similar divided difference relation for $\mathbb{E}\left(A(T)^m\right)$, described in the next section, but we now complete our derivation of Theorem 2.1.

Proof of Theorem 2.1. Applying (1.4, 1.5, 2.5) and (4.1), we obtain

$$\begin{aligned} \mathbb{E}S(T)A(T) - \mathbb{E}S(T)\mathbb{E}A(T) &= \exp[rT, (2r+\sigma^2)T] - e^{rT}(e^{rT}-1)/(rT) \\ &= \exp[rT, (2r+\sigma^2)T] - \exp[rT, 2rT] \\ &= \sigma^2 T \exp[rT, 2rT, (2r+\sigma^2)T]. \qquad (2.9) \end{aligned}$$

Further,

$$\operatorname{var} S(T) = \mathbb{E}(S(T)^2) - (\mathbb{E}S(T))^2 = e^{(2r+\sigma^2)T} - e^{2rT} = \sigma^2 T \exp[2rT, (2r+\sigma^2)T], \qquad (2.10)$$

and, by (1.5, 2.6),

$$\begin{aligned} \operatorname{var} A(T) &= 2\exp[0, rT, (2r+\sigma^2)T] - \left(\frac{e^{rT}-1}{rT}\right)^2 \\ &= 2\exp[0, rT, (2r+\sigma^2)T] - 2\exp[0, rT, 2rT] \\ &= 2\sigma^2 T \exp[0, rT, 2rT, (2r+\sigma^2)T], \qquad (2.11) \end{aligned}$$

using the divided difference recurrence (4.1) once more. Hence

$$R = \frac{\exp[rT, 2rT, (2r+\sigma^2)T]}{\sqrt{2\exp[2rT, (2r+\sigma^2)T]\exp[0, rT, 2rT, (2r+\sigma^2)T]}}. \qquad (2.12)$$

□

It is remarkable that the divided differences appearing in (2.12) are coefficients of the cubic polynomial interpolating the exponential function at $0, rT, 2rT, (2r+\sigma^2)T$. We make three further observations:

(1) Armed with an analytic expression for the correlation coefficient, we can apply the exchange option valuation formula of Margrabe (1978) to derive the values of certain Asian options, if we are willing to accept that the time–average is suitably approximated by exponential Brownian motion. We are investigating the numerics of this rather simple approximation at present and preliminary results are surprisingly promising.

(2) The correlation coefficient $R(rT, \sigma^2 T)$ is typically close to unity: typical values of r, σ and T produce values of R in the $0.8-0.9$ range. In fact, we are able to prove that the correlation coefficient satisfies $R(rT, \sigma^2 T) \ge 1/\sqrt{2}$, for all $r \ge 0$, $\sigma \ge 0$ and $T > 0$, a surprisingly high lower bound for the correlation coefficient. The details of this derivation are too complicated to include here, and we refer the reader to Baxter and Fretwell (2007) for further details. However, the numerical findings are summarised in Figure 1, which displays values of the closely related quantity

$$S \equiv S(r, a) = \frac{(\exp[a, 2r, r])^2}{\exp[a, 2r]\exp[a, 2r, r, 0]},$$

for $-20 \le a \le 40$ and $0.1 \le r \le 10$. It is easily checked that $R(rT, \sigma^2 T) = \sqrt{S(rT, (2r+\sigma^2)T)/2}$, so that the lower bound $R \ge 1/\sqrt{2}$ becomes $S \ge 1$. It is plausible that $S(r, a)$ should be a decreasing function of a, for fixed r, because correlation should be a decreasing function of volatility. Further, it is not difficult to establish the limiting values $\lim_{a \to -\infty} S(a, r) = 2$ and $\lim_{r \to \infty} S(a, r) = 1$. However, further analysis is not straightforward, and the analysis of Baxter and Fretwell (2007) makes great use of properties of divided differences.

(3) It is natural to ask whether these divided difference expressions are particular to exponential Brownian motion. In fact, similar expressions occur when exponential Brownian motion is replaced by certain Lévy-stable variants: see Baxter, Cartea and Fretwell (2007) for further details.

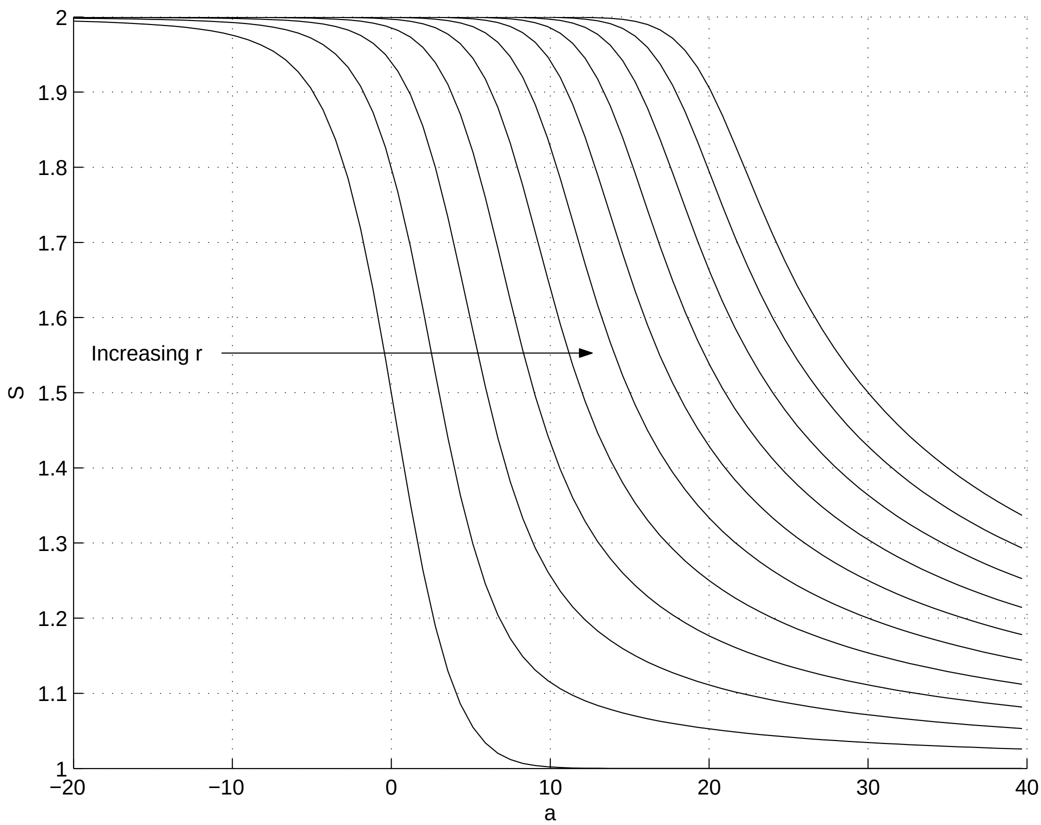

FIGURE 1. $S(r,a)$ for $-20 \le a \le 40$ and $0.1 \le r \le 10$

3. COMPUTING HIGHER MOMENTS OF $A(T)$

We now demonstrate that the neat divided difference formulae obtained for the first and second moments of $A(T)$ are *not* coincidences, but part of a greater pattern from which arise new formulae generalising the moment calculations of Oshanin et al. (1993), Yor (1992) and Yor (2001).

We begin with the iterated integral

$$\mathbb{E}A(T)^m = T^{-m} \int_0^T d\tau_m \int_0^T d\tau_{m-1} \cdots \int_0^T d\tau_1 \; \mathbb{E}S(\tau_1)\cdots S(\tau_m). \tag{3.1}$$

Now, given any point $(\tau_1, \ldots, \tau_m) \in [0,T]^m$, let us sort its components into increasing order, obtaining $(t_1, \ldots, t_n)$ (say). Then

$$\mathbb{E}S(\tau_1)\cdots S(\tau_m) = \mathbb{E}S(t_1)\cdots S(t_m)$$

and

$$\mathbb{E}A(T)^m = m!T^{-m} \int_0^T dt_m \int_0^{t_m} dt_{m-1} \cdots \int_0^{t_2} dt_1 \mathbb{E}S(t_1)\cdots S(t_m). \tag{3.2}$$

Our first task is to calculate the integrand, which we complete after a simple lemma.

Lemma 3.1. *For any positive integer k, we have*

$$\mathbb{E}\left[S(t)^k\right] = \exp\Big(krt + \frac{\sigma^2 t}{2}k(k-1)\Big). \tag{3.3}$$

Proof. This is almost immediate from (1.4):

$$\mathbb{E}S(t)^k = \mathbb{E}e^{k(r-\sigma^2)t+\sigma k\sqrt{t}Z} = e^{k(r-\sigma^2/2)t+\sigma^2k^2t/2} = e^{krt+\sigma^2tk(k-1)/2},$$

where $Z \sim N(0,1)$. □

Proposition 3.2. *If* $0 \le t_1 \le t_2 \le \cdots \le t_m$*, then*

$$\mathbb{E}S(t_1)S(t_2)\cdots S(t_m) = \exp\Big(\sum_{k=1}^{m}\left(r+(m-k)\sigma^2\right)t_k\Big). \tag{3.4}$$

Proof. Lemma 2.2 comprises the case $m = 2$. We complete the proof by induction on the number of terms m, first observing that, by a standard property of geometric Brownian motion,

$$\mathbb{E}S(t_1)S(t_2)\cdots S(t_m) = \mathbb{E}S(t_1)^m\mathbb{E}S(t_2-t_1)\cdots S(t_m-t_1). \tag{3.5}$$

Applying Lemma 3.1 and our induction hypothesis, we obtain

$$\mathbb{E}S(t_1)S(t_2)\cdots S(t_m) = \exp\Big(mrt_1+\sigma^2t_1m(m-1)/2+\sum_{\ell=2}^{m}\left(r+(m-\ell)\sigma^2\right)(t_\ell-t_1)\Big). \tag{3.6}$$

The t_1 coefficient in the exponent is given by

$$mr-(m-1)r+\sigma^2t_1\left(\frac{1}{2}m(m-1)-\sum_{\ell=1}^{m-2}\ell\right) = r+\sigma^2t_1(m-1),$$

using the elementary fact that $m(m-1)/2 = 1+2+\cdots+m-1$. The coefficients of $t_2,\ldots,t_m$ are as already stated in (3.4). □

Thus the desired integral (3.2) becomes

$$\begin{aligned}\mathbb{E}A(T)^m &= m!T^{-m}\int_0^T dt_m\int_0^{t_m}dt_{m-1}\cdots\int_0^{t_2}dt_1\mathbb{E}S(t_1)\cdots S(t_m)\\ &= m!\int_0^1 dt_m\int_0^{t_{m-1}}\cdots\int_0^{t_2}dt_1\exp(\alpha_1t_1+\cdots\alpha_mt_m),\end{aligned} \tag{3.7}$$

where

$$\alpha_k = \left(r+(m-k)\sigma^2\right)T, \qquad k=1,\ldots,m. \tag{3.8}$$

The integral displayed in (3.7) can now be identified as a divided difference using a variant form of the Hermite–Genocchi integral relation.

Theorem 3.3. *Let*

$$b_k := kr+\sigma^2k(k-1)/2, \qquad k=0,1,\ldots. \tag{3.9}$$

Then

$$\mathbb{E}(A(T))^m = m!\exp[b_0T,b_1T,\ldots,b_mT], \qquad m\ge 0. \tag{3.10}$$

Proof. Apply Corollary 4.5 to (3.7) and (3.8), using $\sum_{k=1}^{j}k = j(j+1)/2$. □

The statement of Theorem 3.3 simplifies when $r = \sigma^2$, for then the drift term in (1.1) vanishes, that is, we consider $S(t) = \exp(\sigma\sqrt{t}B(t))$ alone; this is the special case studied by Oshanin et al. (1993) and Yor (1992), for the formulae grow *much* more complicated without the use of divided differences. Therefore we now demonstrate that our expression agrees with theirs.

Theorem 3.4. *If we set $r = \sigma^2/2$ in Theorem 3.3, then we obtain*

$$\begin{aligned}\mathbb{E}\left[A(T)^m\right] &= m!\exp[0, rT, 2^2 rT, 3^2 rT, \ldots, m^2 rT] \\ &= m! H_{\sqrt{rT}}[-m, \ldots, -1, 0, 1, \ldots, m], \end{aligned} \tag{3.11}$$

where $H_c(x) := \exp(c^2 x^2)$, $x \in \mathbb{R}$, for any positive c.

Proof. We simply set $r = \sigma^2$ in Theorem 3.3 and apply (4.14). □

We can now apply Corollary 4.10 to derive the formula given in equation (14) of Oshanin et al. (1993).

Theorem 3.5. *If we set $r = \sigma^2/2$, then*

$$\mathbb{E}\left[A(T)^m\right] = \left(\frac{\Gamma(m)}{\Gamma(2m)}\right) r^{-m} \left(-\frac{1}{2}(-1)^m \binom{2m}{m} + \sum_{\ell=0}^{m} \binom{2m}{\ell} (-1)^\ell e^{rT(m-\ell)^2}\right). \tag{3.12}$$

Proof. Applying Corollary 4.10 to Theorem 3.4, we obtain

$$\begin{aligned}\mathbb{E}\left[A(T)^m\right] &= \left(\frac{m!}{(2m)!}(rT)^{-m} \sum_{k=0}^{2m} \binom{2m}{k} (-1)^k e^{rT(k-m)^2}\right) \\ &= \left(\frac{\Gamma(m)}{\Gamma(2m)}\right) (rT)^{-m} \left(-\frac{1}{2}(-1)^m \binom{2m}{m} + \sum_{\ell=0}^{m} \binom{2m}{\ell} (-1)^\ell e^{rT(m-\ell)^2}\right), \end{aligned} \tag{3.13}$$

after some straightforward algebraic manipulation. □

If we now replace rT by α and m by j in (3.12), then we obtain equation (14) of Oshanin et al. (1993).

4. Divided difference theory

Most of the properties of divided differences required here can be found in Chapter 5 of Powell (1981). However, proofs of the Hermite–Genocchi integral relation are less easily available in the Anglophone mathematical literature, as is our particular variant of it, although the specialist can find much useful material in the treatise of DeVore and Lorentz (1993). We have therefore provided a derivation for the convenience of the reader. The Hermite–Genocchi formula and its consequences are still very much topics of current research; see, for example, Waldron (1998). Furthermore, the result is better served in other European languages; see, for instance, Gel'fond (1963) for a French translation of a Russian classic, or indeed the original Hermite (1878).

We recall the *divided difference recurrence relation.*

Theorem 4.1.

$$f[a_0, a_1, \ldots, a_n] = \frac{f[a_1, \ldots, a_n] - f[a_0, \ldots, a_{n-1}]}{a_n - a_0}, \tag{4.1}$$

for any distinct complex numbers $a_0, \ldots, a_n$.

Proof. See, for instance, Powell (1981), Theorem 5.3. □

If f is sufficiently differentiable, then we can, of course, define divided differences for coincident points. Further, the elementary relation

$$f[a_0, a_1] = \frac{f(a_1) - f(a_0)}{a_1 - a_0} = \int_0^1 f'((1-t)a_0 + t a_1)\, dt, \qquad \text{when } a_0, a_1 \in \mathbb{R}, \tag{4.2}$$

can be generalised to obtain the *Hermite–Genocchi formula.*

Theorem 4.2 (Hermite–Genocchi)**.** *Let $f \in C^{(n)}(\mathbb{R})$ and let $a_0, a_1, \ldots, a_n$ be (not necessarily distinct) real numbers Then, for $n \geq 1$,*

$$\begin{aligned} f[a_0, a_1, \ldots, a_n] &= \int_{S_n} f^{(n)}(t_0 a_0 + t_1 a_1 + \cdots + t_n a_n)\, dt_1 \cdots dt_n, \\ &= \int_0^1 dt_1 \int_0^{1-t_1} dt_2 \cdots \int_0^{1-\sum_{k=1}^{n-1} t_k} dt_n f^{(n)}(t_0 a_0 + t_1 a_1 + \cdots + t_n a_n) \qquad (4.3) \end{aligned}$$

where the domain of integration is the simplex

$$S_n = \left\{ t = (t_1, t_2, \ldots, t_n) \in \mathbb{R}_+^n \colon \sum_{k=1}^n t_k \leq 1 \right\} \qquad (4.4)$$

and

$$t_0 = 1 - \sum_{k=1}^n t_k.$$

Proof. We shall prove (4.3) by induction on n, observing that

$$\int_{S_1} f'(t_0 a_0 + t_1 a_1)\, dt_1 = \int_0^1 f'(a_0 + t_1(a_1 - a_0))\, dt_1 = \frac{f(a_1) - f(a_0)}{a_1 - a_0} = f[a_0, a_1].$$

To extend the formula to higher-order divided differences, we note that

$$f[a_0, a_1, \ldots, a_n, a_{n+1}] = \frac{f[a_1, a_2, \ldots, a_{n+1}] - f[a_0, a_1, \ldots, a_n]}{a_{n+1} - a_0} = g[a_0, a_{n+1}], \qquad (4.5)$$

where

$$g(x) = f[a_1, \ldots, a_n, x], \qquad x \in \mathbb{R}. \qquad (4.6)$$

Now

$$g(x) = \int_{S_n} f^{(n)}(x t_0 + a_1 t_1 + \cdots + a_n t_n)\, dt_1 \cdots dt_n$$

so that

$$g'(x) = \int_{S_n} t_0 f^{(n+1)}(x t_0 + a_1 t_1 + \cdots + a_n t_n)\, dt_1 \cdots dt_n.$$

Therefore

$$\begin{aligned} f[a_0, a_1, \ldots, a_n, a_{n+1}] &= \int_0^1 d\tau\; g'((1-\tau)a_0 + \tau a_{n+1}) \\ &= \int_0^1 d\tau \int_{S_n} dt_1 \cdots dt_n\; t_0 f^{(n+1)}([(1-\tau)a_0 + \tau a_{n+1}]t_0 + a_1 t_1 + \cdots a_n t_n) \\ &= \int_{S_n} dt_1 \cdots dt_n \int_0^1 d\tau\; t_0 f^{(n+1)}([(1-\tau) t_0 a_0 + \sum_{\ell=1}^n a_\ell t_\ell + \tau t_0 a_{n+1}) \\ &= \int_0^1 dt_1 \int_0^{1-t_1} dt_2 \cdots \int_0^{1-\sum_{k=1}^{n+1} t_k} dt_{n+1} f^{(n+1)} \left(T_0 a_0 + \sum_{k=1}^{n+1} t_k a_k \right) \\ &= \int_{S_{n+1}} f^{(n+1)} \left(T_0 a_0 + t_1 a_1 + \cdots + t_{n+1} a_{n+1} \right)\, dt_1 \cdots dt_{n+1}, \end{aligned}$$

where we have used the substitution $t_{n+1} = t_0 \tau$ and the notation $T_0 = 1 - \sum_{k=1}^{n+1} t_k$.
□

We shall need a variant form of the Hermite–Genocchi integral relation for which the following notation is useful. Given any real $n \times n$ nonsingular matrix V, with columns $v_1, \ldots, v_n$, we let $K(V)$ denote the closed convex hull of $0, v_1, \ldots, v_n$, i.e.

$$K(V) := \operatorname{conv}\{0, v_1, \ldots, v_n\}.$$

In this notation, the Hermite–Gnocchi integral relation states that

$$f[a_0, a_1, \ldots, a_n] = \int_{K(I_n)} f^{(n)}\left(a_0 + (a - a_0 e)^T y\right) dy, \tag{4.7}$$

where

$$a = \begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix}, \qquad e = \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix},$$

and I_n denotes the $n \times n$ identity matrix. Integrating the nth derivative over the simplex $K(V)$ yields a useful variant form of Hermite–Genocchi.

Theorem 4.3. *Let $V \in \mathbb{R}^{n \times n}$ be any nonsingular matrix. Then*

$$\frac{1}{|\det V|} \int_{K(V)} f^{(n)}\left(a^T y\right) dy = f[0, (V^T a)_1, \ldots, (V^T a)_n], \tag{4.8}$$

where $(V^T a)_k$ denotes the kth component of the vector $V^T a$.

Proof. Substituting $y = Vz$, Hermite–Genocchi implies the relation

$$\int_{K(V)} f^{(n)}\left((V^T a)^T z\right) dz = f[0, (V^T a)_1, \ldots, (V^T a)_n].$$

□

Corollary 4.4. *For any function $f \in C^{(n)}(\mathbb{R})$, we have*

$$\begin{aligned} &\int_0^1 dx_n \int_0^{x_n} dx_{n-1} \cdots \int_0^{x_2} dx_1 f^{(n)}\left(\sum_{k=1}^n a_k x_k\right) \\ &\qquad = f[0, a_n, a_n + a_{n-1}, \ldots, a_n + a_{n-1} + \cdots + a_1]. \end{aligned} \tag{4.9}$$

Proof. Set

$$V = \begin{pmatrix} 1 & & & \\ 1 & 1 & & \\ \vdots & & \ddots & \\ 1 & 1 & \cdots & 1 \end{pmatrix}$$

in Theorem 4.3. □

The exponential function is a particularly important case for us, in which case the Hermite–Genocchi formula becomes

$$\exp[a_0, a_1, \ldots, a_n] = \int_{S_n} e^{t_0 a_0 + t_1 a_1 + \cdots + t_n a_n} \, dt_1 \cdots dt_n \tag{4.10}$$

and Corollary 4.4 takes the following form.

Corollary 4.5. *We have*

$$\begin{aligned} &\int_0^1 dx_n \int_0^{x_n} dx_{n-1} \cdots \int_0^{x_2} dx_1 \exp\left(\sum_{k=1}^n a_k x_k\right) \\ &\qquad = \exp[0, a_n, a_n + a_{n-1}, \ldots, a_n + a_{n-1} + \cdots + a_1]. \end{aligned} \tag{4.11}$$

Proof. Let f be the exponential function in Corollary 4.4. □

Further, we note that, for the exponential function, Theorem 4.3 becomes the interesting formula

$$\frac{1}{|\det V|}\int_{K(V)} e^{a^T y}\, dy = \exp[0, (V^T a)_1, \ldots, (V^T a)_n]. \tag{4.12}$$

Thus, integrating exponentials over simplexes or, more generally, a polyhedron formed by the disjoint union of simplexes, will generate divided differences of the exponential.

We shall also need two simple preliminary results. Let us use $\mathbb{P}_n$ to denote the vector space of polynomials of degree n.

Lemma 4.6. *We have*

$$\exp(\mu)\exp[\lambda_0, \ldots, \lambda_m] = \exp[\lambda_0 + \mu, \ldots, \lambda_m + \mu], \tag{4.13}$$

where $\lambda_0, \ldots, \lambda_m$ *and* μ *can be any complex numbers.*

Proof. Immediate. □

Lemma 4.7. *Let* $f\colon \mathbb{C} \to \mathbb{C}$ *and let* $a_1, \ldots, a_n$ *be distinct nonzero complex numbers. Then*

$$f[0, a_1^2, \ldots, a_n^2] = g[-a_n, \ldots, -a_1, 0, a_1, \ldots, a_n], \tag{4.14}$$

where $g(z) = f(z^2)$*, for* $z \in \mathbb{C}$.

Proof. Let $p \in \mathbb{P}_n$ interpolate f at $0, a_1^2, \ldots, a_n^2$. Then $q(z) := p(z^2)$ is a polynomial of degree $2n$ satisfying $q(\pm a_j) = p(a_j^2) = f(a_j^2) = g(\pm a_j)$, for $j = 0, \ldots, n$, setting $a_0 = 0$, for convenience. The result then follows from uniqueness of the interpolating polynomial.

□

It is well-known that a divided difference at equally spaced points can be expressed in a particularly simple form using the forward difference operator

$$\Delta_h f(x) := f(x+h) - f(x),$$

which we shall need when demonstrating the equivalence between our moment calculations and those of Oshanin et al. (1993) and Yor (1992). The next proposition is well-known, but we again include its short proof for the reader's convenience.

Proposition 4.8. *Let* $f\colon \mathbb{R} \to \mathbb{R}$*, let* h *be any positive constant and let* n *be a non-negative integer. Then*

$$f[x, x+h, x+2h, \ldots, x+nh] = \frac{\Delta_h^n f(x)}{n!h^n}. \tag{4.15}$$

Proof. It is easily checked that $f[x, x+h] = \Delta_h f(x)/h$. Further, if we assume (4.15) for $n-1$, then the divided difference recurrence relation implies that

$$\begin{aligned}
&f[x, x+h, \ldots, x+nh] \\
&= \frac{f[x+h, \ldots, x+nh] - f[x, x+h, \ldots, x+(n-1)h]}{nh} \\
&= \frac{\Delta_h f[x, \ldots, x+(n-1)h]}{nh} \\
&= \frac{1}{nh}\Delta_h\left(\frac{\Delta_h^{n-1} f(x)}{(n-1)!h^{n-1}}\right) \\
&= \frac{\Delta_h^n f(x)}{n!h^n}.
\end{aligned}$$

Thus the result follows by induction. ∎

Corollary 4.9. *Let $f\colon \mathbb{R} \to \mathbb{R}$ and let h be any positive constant. Then*

$$f[x, x+h, x+2h, \ldots, x+nh] = \frac{1}{n!h^n} \sum_{k=0}^{n} \binom{n}{k} (-1)^{n-k} f(x+kh). \tag{4.16}$$

Proof. We define the *forward shift* operator

$$E_h f(x) := f(x+h), \qquad x \in \mathbb{R},$$

and observe that, by the binomial theorem,

$$\Delta_h^n f(x) = (E_h - 1)^n f(x) = \sum_{k=0}^{n} \binom{n}{k} (-1)^{n-k} E_h^k f(x) = \sum_{k=0}^{n} \binom{n}{k} (-1)^{n-k} f(x+kh).$$

∎

Corollary 4.10. *Let $f\colon \mathbb{R} \to \mathbb{R}$ and let h be any positive number. Then*

$$f[-nh, -(n-1)h, \ldots, -h, 0, h, \ldots, nh] = \frac{1}{(2n)!h^{2n}} \sum_{k=0}^{2n} \binom{2n}{k} (-1)^k f((k-n)h). \tag{4.17}$$

Proof. This is an immediate consequence of Corollary 4.9. ∎

We shall also need the *Leibniz* relation for divided differences of a product when deriving the recurrence differential equation for moments.

Theorem 4.11 (Leibniz)**.** *Let D be any subset of $\mathbb{C}$ containing the distinct points $z_0, z_1, \ldots, z_n$ and let v and w be complex-valued functions on D. If $u = v \cdot w$, then*

$$u[z_0, \ldots, z_n] = \sum_{k=0}^{n} v[z_0, \ldots, z_k] w[z_k, \ldots, z_n]. \tag{4.18}$$

Proof. Let $p \in \mathbb{P}_n$ be the unique polynomial interpolant for u written in standard Newton form, that is,

$$p(z) = v[z_0] + v[z_0, z_1](z - z_0) + \cdots v[z_0, z_1, \ldots, z_n](z - z_0) \cdots (z - z_{n-1}). \tag{4.19}$$

We shall let $q \in \mathbb{P}_n$ be the unique polynomial interpolating w, but with the points chosen in the order $z_n, z_{n-1}, \ldots, z_0$, that is,

$$q(z) = w[z_n] + w[z_n, z_{n-1}](z - z_n) + \cdots + w[z_n, \ldots, z_0](z - z_n) \cdots (z - z_1). \tag{4.20}$$

Now their product $p \cdot q$ is a polynomial of degree $2n$. Dividing this polynomial by $(z - z_0) \cdots (z - z_n)$, we obtain

$$p(z)q(z) = r(z) + s(z)(z - z_0) \cdots (z - z_n),$$

where $r \in \mathbb{P}_n$. We see that $u(z_j) = v(z_j)w(z_j) = p(z_j)q(z_j) = r(z_j)$, for $0 \le j \le n$. Hence, by uniqueness of the polynomial interpolant for u in $\mathbb{P}_n$, we obtain

$$r(z) = u[z_0] + \cdots + u[z_0, \ldots, z_n](z - z_0) \cdots (z - z_n). \tag{4.21}$$

We obtain (4.18) by equating the coefficients of z^n in (4.21) and the product of the expressions in (4.19) and (4.20), modulo $(z - z_0) \cdots (z - z_n)$. ∎

5. Towards the correlation coefficient bound

Let us define

$$R_m(\alpha) = e^\alpha - \sum_{k=0}^{m} \frac{\alpha^k}{k!}, \tag{5.1}$$

for all non-negative integer m and $\alpha \in \mathbb{R}$; thus $R_m(\alpha)$ is the exponential function Taylor remainder. Then we can also write the remainder as

$$R_m(\alpha) = \alpha^{m+1} \exp[\underbrace{0, 0, \ldots, 0}_{m+1}, \alpha]. \tag{5.2}$$

Furthermore,

$$R'_m(\alpha) = R_{m-1}(\alpha), \qquad \text{for } m \geq 1, \alpha \in \mathbb{R}. \tag{5.3}$$

Lemma 5.1. *The exponential function Taylor remainders satisfy*

$$\frac{R_{m+1}}{R_m(\alpha)} = 1 - \frac{1}{(m+1)! \exp[\underbrace{0, 0, \ldots, 0}_{m+1}, \alpha]}. \tag{5.4}$$

Proof. We have

$$\begin{aligned} 1 - \frac{R_{m+1}(\alpha)}{R_m(\alpha)} &= \frac{R_m(\alpha) - R_{m+1}(\alpha)}{R_m(\alpha)} \\ &= \frac{p_{m+1}(\alpha) - p_m(\alpha)}{R_m(\alpha)} \\ &= \frac{\alpha^{m+1}}{((m+1)! R_m(\alpha)} \\ &= \frac{1}{(m+1)! \exp[\underbrace{0, 0, \ldots, 0}_{m+1}, \alpha]}, \end{aligned}$$

using (5.1) and (5.2). □

Lemma 5.2. *The function* $\exp[\underbrace{0, 0, \ldots, 0}_{m+1}, \alpha]$*, for* $\alpha \in \mathbb{R}$*, is an increasing function, with derivative*

$$\exp[\underbrace{0, 0, \ldots, 0}_{m+1}, \alpha, \alpha]. \tag{5.5}$$

Proof. Now the divided difference recurrence relation implies the relation

$$\exp[\underbrace{0, 0, \ldots, 0}_{m+1}, \alpha + h] - \exp[\underbrace{0, 0, \ldots, 0}_{m+1}, \alpha] = \exp[\underbrace{0, 0, \ldots, 0}_{m+1}, \alpha, \alpha + h],$$

which is strictly positive for positive h. Thus $\exp[\underbrace{0, 0, \ldots, 0}_{m+1}, \alpha]$ is an creasing function, for $\alpha \in \mathbb{R}$, and we have also shown that

$$\frac{d}{d\alpha} \exp[\underbrace{0, 0, \ldots, 0}_{m+1}, \alpha] = \exp[\underbrace{0, 0, \ldots, 0}_{m+1}, \alpha, \alpha].$$

□

Corollary 5.3. *The function* $R_{m+1}(\alpha)/R_m(\alpha)$*, for* $\alpha \in \mathbb{R}$*, is an increasing function.*

Proof. This is immediate from Lemmata 5.1 and 5.2, since

$$\frac{d}{d\alpha}\frac{R_{m+1}(\alpha)}{R_m(\alpha)} = \frac{\exp[\underbrace{0,0,\ldots,0}_{m+1},\alpha,\alpha]}{(m+1)!\exp[\underbrace{0,0,\ldots,0}_{m+1},\alpha]^2}.$$

□

Corollary 5.4. *We have the inequality*

$$R_m(\alpha)^2 \geq R_{m+1}(\alpha)R_{m-1}(\alpha), \qquad \text{for } m \geq 1 \text{ and } \alpha \in \mathbb{R}.$$

Proof. The non-negative derivative of $R_{m+1}(\alpha)/R_m(\alpha)$ can also be written in the form

$$0 \leq \frac{d}{d\alpha}\frac{R_{m+1}(\alpha)}{R_m(\alpha)} = \frac{R_m(\alpha)^2 - R_{m+1}(\alpha)R_{m-1}(\alpha)}{R_m(\alpha)^2}.$$

□

Proposition 5.5. *Define*

$$Z_m(\alpha) = \exp[\underbrace{0,0,\ldots,0}_{m+1},\alpha], \tag{5.6}$$

for non-negative integer m and $\alpha \in \mathbb{R}$. Then $\{Z_m(\alpha)\}$ is a log-concave sequence, that is,

$$Z_m(\alpha)^2 \geq Z_{m+1}(\alpha)Z_{m-1}(\alpha), \qquad \text{for } m \geq 1. \tag{5.7}$$

Proof. We need only note that $Z_m(\alpha) = \alpha^{-m-1}R_m(\alpha)$, so that

$$\frac{Z_m(\alpha)}{Z_{m-1}(\alpha)} = \alpha^{-1}\frac{R_m(\alpha)}{R_{m-1}(\alpha)}. \tag{5.8}$$

□

Similar expressions occur for the general case. If we let $p_m(\alpha)$ be the unique polynomial of degree m interpolating the exponential function at $\alpha_0, \alpha_1, \ldots, \alpha_m$, then the polynomial remainder $S_m(\alpha) := \exp(\alpha) - p_m(\alpha)$ satisfies

$$S_m(\alpha) = \exp[\alpha_0,\alpha_1,\ldots,\alpha_m,\alpha]\prod_{\ell=0}^{m}(\alpha-\alpha_\ell). \tag{5.9}$$

Hence the result analogous to Lemma 5.1 is as follows.

Lemma 5.6. *We have the relation*

$$\frac{S_{m+1}(\alpha)}{S_m(\alpha)} = 1 - \frac{\exp[\alpha_0,\ldots,\alpha_m,\alpha_{m+1}]}{\exp[\alpha_0,\ldots,\alpha_m,\alpha]}. \tag{5.10}$$

Proof. By direct calculation,

$$
\begin{aligned}
1 - \frac{S_{m+1}(\alpha)}{S_m(\alpha)} &= \frac{S_m(\alpha) - S_{m+1}(\alpha)}{S_m(\alpha)} \\
&= \frac{p_{m+1}(\alpha) - p_m(\alpha)}{S_m(\alpha)} \\
&= \frac{\exp[\alpha_0, \ldots, \alpha_m, \alpha_{m+1}] \prod_{\ell=0}^m (\alpha - \alpha_\ell)}{\exp[\alpha_0, \ldots, \alpha_m, \alpha] \prod_{\ell=0}^m (\alpha - \alpha_\ell)} \\
&= \frac{\exp[\alpha_0, \ldots, \alpha_m, \alpha_{m+1}]}{\exp[\alpha_0, \ldots, \alpha_m, \alpha]}
\end{aligned}
$$

□

6. A recurrence relation

The Feynman–Kac formula (see, for example, Karatzas and Shreve (1991)) suggests that the moments $E_n(t) := \mathbb{E}\left(A(t)^n\right)$ of the time average should satisfy a certain differential equation, which we shall also obtain as an illustration of the divided difference approach.

Theorem 6.1. *Let $\{c_n\}_{n=1}^\infty$ be any strictly increasing sequence of positive numbers and define $e_n : (0, \infty) \to \mathbb{R}$ by the divided difference*

$$e_n(t) = \exp[0, c_1 t, \ldots, c_n t], \qquad t > 0, \quad n \geq 0. \tag{6.1}$$

Then

$$t e_n'(t) = e_n(t)\,(c_n t - n) + e_{n-1}(t), \qquad \textit{for } n \geq 1. \tag{6.2}$$

Proof. Applying the Hermite–Genocchi formula, we obtain

$$e_n(t) = \int_{K(I_n)} \exp\left(t c^T y\right)\, dy, \tag{6.3}$$

where $b = (c_1, \ldots, c_n)^T$, and differentiating (6.3) yields

$$e_n'(t) = \int_{K(I_n)} \exp(t c^T y)(c^T y)\, dy. \tag{6.4}$$

Now writing $g(s) = s$ and applying Leibniz's formula for divided differences, we find

$$
\begin{aligned}
(g \cdot \exp)\,[0, c_1 t, \ldots, c_n t] &= g[0] \exp[0, c_1 t, \ldots, c_n t] + g[0, c_1 t] \exp[c_1 t, \ldots, c_n t] \\
&= \exp[c_1 t, \ldots, c_n t]. \qquad (6.5)
\end{aligned}
$$

Further, the relation $(g \cdot \exp)^{(n)} = g \cdot \exp + n \exp$ and (6.5) imply

$$
\begin{aligned}
t e_n'(t) &= \int_{K(I_n)} (g \cdot \exp)^{(n)}\,(t c^T y)\, dy - n \int_{K(I_n)} \exp(t c^T y)\, dy \\
&= (g \cdot \exp)\,[0, c_1 t, \ldots, c_n t] - n \exp[0, c_1 t, \ldots, c_n t] \\
&= \exp[c_1 t, \ldots, c_n t] - n e_n(t). \qquad (6.6)
\end{aligned}
$$

However,

$$e_n(t) = \frac{\exp[c_1 t, \ldots, c_n t] - e_{n-1}(t)}{c_n t}, \tag{6.7}$$

by the divided difference recurrence relation, so that

$$\exp[c_1 t, \ldots, c_n t] = c_n t e_n(t) + e_{n-1}(t). \tag{6.8}$$

Substituting (6.8) in (6.6) provides (6.2). □

The corresponding differential equation for E_n is now immediate.

Corollary 6.2. *The moments satisfy*

$$tE_n'(t) = E_n(t)\,(b_n t - n) + E_{n-1}(t), \qquad \text{for } n \geq 1, \tag{6.9}$$

where b_n *is given by* (3.9).

Proof. We apply Theorem 6.1 and Theorem 3.3. □

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