On approximation by exponentials

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Abstract We consider an approximation of $L^2[0,\infty)$ functions by linear combinations of exponentials $\{\exp(-\lambda_\ell t)\}$. Having derived explicitly by Fourier transform techniques an orthogonal basis of exponentials, we specialize the discussion to the choice $\lambda_\ell = q^\ell$, $\ell = 0, 1, \ldots$, where $q \in (0,1)$. In that case the underlying orthogonal functions possess a particularly appealing form and they obey interesting recurrence relations. We conclude the paper with a brief discussion of convergence issues.

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1. Orthogonality by sums of exponentials

Let λ be a complex number with positive real part and define the function $g_\lambda: \mathbb{R} \to \mathbb{R}$ by

$$g_\lambda(x) = \begin{cases} e^{-\lambda x}, & x \geq 0, \\ 0, & x < 0 \end{cases} \tag{1.1}$$

Thus g_λ is a member of the Hilbert space $H := \{f \in L^2(\mathbb{R}^2) : \operatorname{supp} f \subset [0, \infty)\}$, endowed with the usual inner product. Further, for any $f \in H$ we have the useful relation

$$(f, g_\lambda) = \hat{f}(-i\lambda), \tag{1.2}$$

where we have used the well known fact (see Dym and McKean (1972)) that the Fourier transform of any member of H can be analytically continued throughout the lower half plane $\{z \in \mathbb{C} : \operatorname{Im} z < 0\}$. We shall also need the Fourier transform of g_λ:

$$\hat{g}_\lambda(z) = \frac{-i}{z - i\lambda}. \tag{1.3}$$

Now let $(\lambda_k)_{k=0}^\infty$ be any sequence of complex numbers with positive real parts. Our aim is to study the subspaces

$$H_n = \text{ span } \{g_{\lambda_0}, g_{\lambda_1}, \ldots, g_{\lambda_n}\} \tag{1.4}$$

The simple form of (1.3) allows us to construct orthogonal bases for these subspaces.

Theorem 1 *$\hat{H}_n \ominus \hat{H}_{n-1}$ is the one-dimensional subspace generated by the function*

$$\widehat{r}_n(z) = \frac{-i}{z - i\lambda_n} \prod_{k=0}^{n-1} \frac{z + i\lambda_k}{z - i\lambda_k}. \tag{1.5}$$

Proof Every element of $\hat{H}_n \ominus \hat{H}_{n-1}$ is a linear combination of the rational functions $\{(z - i\lambda_k)^{-1} : 0 \leq k \leq n\}$; this defines $\hat{r}_n$ up to multiplication by a constant. □

Partial fraction decomposition of the rational function (1.5) yields the coefficients of r_n when expressed as a linear combination of $g_{\lambda_0}, \ldots, g_{\lambda_n}$.

Proposition 2 *We have*

$$r_n(t) = \sum_{j=0}^{n} \exp(-\lambda_j t) \frac{\prod_{k=0}^{n-1}(\lambda_j + \lambda_k)}{\prod_{\ell=0, \ell \neq j}^{n}(\lambda_j - \lambda_\ell)}. \tag{1.6}$$

Proof Writing $\hat{r}_n(z) = -i \sum_{k=0}^{n} \alpha_k (z - i\lambda_k)^{-1}$, we obtain

$$-i \sum_{k=0}^{n} \alpha_k \prod_{\ell=0, \ell \neq k}^{n} (z - i\lambda_\ell) = -i \prod_{k=0}^{n-1} (z + i\lambda_k).$$

Setting $z = \mathrm{i}\lambda_j$ provides the relation

$$\alpha_j \prod_{\ell=0,\ell\neq j}^{n} (\lambda_j - \lambda_\ell) = \prod_{k=0}^{n-1} (\lambda_j + \lambda_k),$$

whence the result. □

The formula derived in (1.6) is easily recognized to be a certain divided difference.

Proposition 3 *Let* $s_n(z,t) = \exp(-zt) \prod_{k=0}^{n-1}(z+\lambda_k)$. *Then*

$$r_n(t) = s_n(\,\cdot\,,t)[\lambda_0, \lambda_1, \ldots, \lambda_n]. \tag{1.7}$$

Proof The standard algebraic identity

$$f[\lambda_0, \ldots, \lambda_n] = \sum_{j=0}^{n} \frac{f(\lambda_j)}{\prod_{\ell=0,\ell\neq j}^{n} (\lambda_j - \lambda_\ell)}$$

is valid for every function f defined at the points $\lambda_0, \ldots, \lambda_n$. □

As a consequence of this divided difference relation, we find that the normalization chosen for r_n implies the equation $r_n(0) = 1$.

Corollary 4 *For every* n, *we have* $r_n(0) = 1$.

Proof When $t = 0$, the function $z \mapsto s_n(z, 0)$ is a monic polynomial of degree n. Hence the divided difference appearing on the right hand side of (1.7) is equal to one for any choice of $\lambda_0, \ldots, \lambda_n$.

Corollary 5 *The two-norm of* r_n *is given by* $\|r_n\|^2 = (2\lambda_n)^{-1}$.

Proof The Parseval Theorem provides the equation

$$\|r_n\|^2 = (2\pi)^{-1} \int_{\mathbb{R}} |\hat{r}_n(z)|^2 \, \mathrm{d}z.$$

However, for real x we have $|\hat{r}_n(x)|^2 = (x^2 + \lambda_n^2)^{-1}$. Therefore we need only compute the elementary integral

$$\|r_n\|^2 = (2\pi)^{-1} \int_{\mathbb{R}} (x^2 + \lambda_n^2)^{-1} \, \mathrm{d}x = (2\lambda_n)^{-1}.$$

□

The complex analytic theory of the Müntz theorem is closely related to the material discussed here. For example, if $(\lambda_k)_{k=0}^{\infty}$ is a sequence of positive numbers possessing a convergent subsequence with positive limit, then linear combinations of the functions $g_{\lambda_0}, g_{\lambda_1}, \ldots$ are dense in H. For, suppose $f \in H$ were orthogonal to these functions. The the Fourier transform $\hat{f}$ satisfies $\hat{f}(-\mathrm{i}\lambda_k) = 0$ for all k, and, by the principle of isolated zeros for analytic functions, must therefore vanish identically. The Parseval theorem

$$\int_{\mathbb{R}} |f(x)|^2 \,\mathrm{d}x = (2\pi)^{-1} \int_{\mathbb{R}} |\hat{f}(z)|^2 \,\mathrm{d}z$$

then implies that f vanishes almost everywhere. The particular choice $\lambda = \alpha + q^k$, where $0 < q < 1$ and $\alpha > 0$, obviously accumulates at the point α, and this particular case is extensively studied below.

The proof strategy of the preceding paragraph yields a derivation of Lerch's uniqueness theorem for Laplace tranforms that deserves to be better known.

Theorem 6 *Let $f\colon [0,\infty) \to \mathbb{R}$ be a measurable function for which*

$$\int_0^\infty \exp(-s_0 t)|f(t)| \,\mathrm{d}t < \infty$$

for some $s_0 \geq 0$, and the Laplace transform

$$\int_0^\infty \mathrm{e}^{-st} f(t) \,\mathrm{d}t = 0, \tag{1.8}$$

for all sufficiently large s. Then f vanishes almost everywhere.

Proof The function $g(t) := \exp(-s_0 t) f(t)$ is absolutely integrable. Hence the dominated convergence theorem implies the continuity of its Fourier transform

$$\hat{g}(z) = \int_0^\infty \mathrm{e}^{-izt} g(t) \,\mathrm{d}t$$

for $\mathrm{Im}\, z < 0$. Applying Morera's theorem, we deduce that the Fourier transform is, in fact, analytic for $\mathrm{Im}\, z < 0$. However, 1.8 implies that $\hat{g}$ vanishes on an infinite subinterval of the imaginary axis, and must therefore vanish everywhere by the principle of isolated zeros. Hence f vanishes almost everywhere. □

We have not seen this proof in the literature, but its novelty is implausible.

The Müntz theorem is often proved using Cauchy's determinant identity; see, for instance, Lemma 11.3.1 of Davis (1975). It is noteworthy that our construction of r_n enables us to bypass this identity in an illuminating way. Specifically, let $g_p(t) = \exp(-pt)$, where p is not one of the numbers in the sequence $\{\lambda_k\}$. Then we can explicitly determine the distance from g_p to H_n.

Proposition 7 *We have*

$$\mathrm{dist}(g_p, H_n)^2 = (2p)^{-1} \left(\prod_{k=0}^{n} \frac{p - \lambda_k}{p + \lambda_k} \right)^2 .$$

Proof Replacing λ_{n+1} by p *pro tem* in Theorem 1, we see that the closest function $f_n \in H_n$ to g_p must satisfy the relation

$$\hat{g}_p(z) - \hat{f}_n(z) = \frac{-\mathrm{i}c}{z - \mathrm{i}p} \prod_{k=0}^{n} \frac{z + \mathrm{i}\lambda_k}{z - \mathrm{i}\lambda_k},$$

where the constant c is chosen so that the coefficient of g_p is unity. Applying (1.6), we conclude that

$$c = \prod_{k=0}^{n} \frac{p - \lambda_k}{p + \lambda_k},$$

and Corollary 5 completes the proof. □

It is an interesting elementary exercise to deduce the Cauchy determinant identity from Proposition 7. To deduce one half of the Müntz theorem, let $0 < \lambda_0 < \lambda_1 < \cdots$ be any sequence for which $\lambda_k \to \infty$ and $\sum \lambda_k^{-1}$ is finite. Then the infinite product

$$\Delta(z) = \prod_{k=0}^{\infty} \frac{1 - z/\lambda_k}{1 + z/\lambda_k} \tag{1.9}$$

is absolutely convergent. Thus Δ is an analytic function in, say, the domain $C \setminus (-\infty, 0]$ whose only zeros are located at the points $\{\lambda_k\}$. But we have the relation

$$\lim_{n \to \infty} 2p \, \mathrm{dist}(g_n, H_n) = \Delta(p)^2 > 0.$$

We conclude this section by presenting a universal differential recurrence relation which is obeyed by members of the sequence $\{r_n\}_n^\infty$. Our point of departure is (1.5), which immediately implies that

$$(\mathrm{i}z + \lambda_n)\hat{r}_n(z) = (\mathrm{i}z - \lambda_{n-1})\hat{r}_{n-1}(z), \tag{1.10}$$

whence

$$\widehat{r'}_{n-1} - \widehat{r'}_{n-1} = \lambda_{n-1}\hat{r}_{n-1} + \lambda_n \hat{r}_n.$$

Therefore, since the Fourier transform is an isometric linear isomorphism on $L^2(\mathbb{R})$, and by virtue of the analyticity of each r_n, we deduce

Theorem 8 *The sequence $r_0, r_1, \ldots$ obeys the differential recurrence relation*

$$r'_{n-1} - r'_n = \lambda_{n-1} r_{n-1} + \lambda_n r_n, \qquad n = 1, 2, \ldots. \tag{1.11}$$

□

Formula (1.11) can be recast into an interesting form. We commence by noting that (1.5) implies the recursion

$$\begin{aligned} \hat{r}_n(z) &= \left(\frac{z + \mathrm{i}\lambda_{n-1}}{z - \mathrm{i}\lambda_n} \right) \hat{r}_{n-1}(z) \\ &= \hat{r}_{n-1}(z) - (\lambda_{n-1} + \lambda_n)\widehat{g_{\lambda_n} * r}_{n-1}(z). \end{aligned}$$

Thus

$$r_n(t) = r_{n-1}(t) - (\lambda_{n-1} + \lambda_n) \int_0^t \mathrm{e}^{-\lambda_n (t-\tau)} r_{n_1}(\tau) \, \mathrm{d}\tau. \tag{1.12}$$

2. Approximation by exponentials with rescaling

A particularly appealing choice of parameters is $\lambda_k = q^k + \alpha$, where $q \in (0,1)$ and α is positive. According to Section 1, the functions $\exp(-q^k x - \alpha)$, $k = 0, 1, \ldots$, are dense with respect to the inner product

$$\langle f, g \rangle = \int_0^\infty f(x) g(x) \, \mathrm{d}x$$

or, alternatively, $\exp(-q^k x)$, $k = 0, 1, \ldots$, are dense with respect to

$$\langle f, g \rangle_\alpha = \int_0^\infty f(x) g(x) \mathrm{e}^{-2\alpha x} \, \mathrm{d}x. \tag{2.1}$$

Moreover, the linear spaces H_n which have been defined in (1.4) are closed under shifts

$$f \in H_n \qquad \Longrightarrow \qquad f(\,\cdot\, + \beta) \in H_n \quad \text{for all } \beta \in \mathbb{R};$$

whilst a dilation of the independent variable by a factor of q moves up the chain $H_0 \subset H_1 \subset H_2 \subset \cdots$

$$f \in H_n \qquad \Longrightarrow \qquad f(q\,\cdot\,) \in H_{n+1}.$$

We use (1.6) to describe an orthogonal basis of $+_{n=0}^{\infty} H_n$ in a closed form,

$$r_m^{[\alpha]}(t) = \mathrm{e}^{-\alpha t} \sum_{j=0}^{m} \frac{\prod_{k=0}^{m-1} (q^k + q^j + 2\alpha)}{\prod_{\ell=0, \ell \neq j}^{m} (q^j - q^\ell)} \mathrm{e}^{-q^j t}, \qquad m = 0, 1, \ldots. \tag{2.2}$$

The last expression can be somewhat simplified by using Gauß–Heine symbols but this procedure has not led to significant additional insight. However, a substantially simplified form, accompanied by a wealth of further results – generating functions, recurrence relations, connections to certain functional-differential equations – follows in the case $\alpha = 0$. The *quid pro quo* is, of course, that density with respect to the inner product (2.2) is lost. Although it is not difficult to prove that density is retained for certain subspaces of $L^2[0, \infty)$, we prefer to concern ourselves with the rich class of relations satisfied when $\alpha = 0$.

Recall that the *Gauß–Heine symbol,* also known as the *q-factorial* (Gasper & Rahman, 1990), reads

$$(z;q)_0 = 1, \qquad (z;q)_n = (1 - q^{n-1} z)(z;q)_{n-1} = \prod_{j=0}^{n-1} (1 - q^j z), \quad j = 1, 2, \ldots.$$

Since

$$\begin{aligned} \prod_{k=0}^{j-1} (q^j + q^k) &= q^{(j-1)j/2} (-q;q)_j, \\ \prod_{k=j}^{m-1} (q^j + q^k) &= q^{(m-j)j} (-1;q)_{m-j}, \end{aligned}$$

$$\prod_{\ell=0}^{j-1}(q^j - q^\ell) = (-1)^j q^{(j-1)j/2}(q;q)_j, \qquad \text{and}$$

$$\prod_{\ell=j+1}^{m}(q^j - q^\ell) = q^{(m-j)j}(q;q)_j,$$

substitution into (2.2) results in the explicit form

$$R_m(t) := r_{m+1}^{[0]}(t) = \sum_{j=0}^{m}(-1)^j \frac{(-q;q)_j(-1;q)_{m-j}}{(q;q)_j(q;q)_{m-j}} \mathrm{e}^{-q^j t}, \qquad m = 0,1,\ldots. \tag{2.3}$$

Let

$$G(t,z) := \sum_{m=0}^{\infty} R_m(t) z^m, \qquad |z| < 1.$$

Proposition 9 *The function G obeys the functional-differential equation*

$$\frac{\partial}{\partial t}G(t,z) + G(t,z) = -z\left[G(qt,z) - q\frac{\partial}{\partial t}G(qt,z)\right] \tag{2.4}$$

with the initial condition

$$G(0,z) = \frac{1}{1-z}. \tag{2.5}$$

Proof We multiply (2.3) by z^m and sum. Interchanging the order of summation, we obtain

$$\begin{aligned} G(t,z) &= \sum_{m=0}^{\infty}\sum_{j=0}^{m}(-1)^j \frac{(-q;q)_j(-1;q)_{m-j}}{(q;q)_j(q;q)_{m-j}} \mathrm{e}^{-q^j t} z^m \\ &= \sum_{j=0}^{\infty}(-1)^j \frac{(-q;q)_j}{(q;q)_j} \mathrm{e}^{-q^j t} \sum_{m=j}^{\infty} \frac{(-1;q)_{m-j}}{(q;q)_{m-j}} z^m \\ &= \left[\sum_{j=0}^{\infty}(-1)^j \frac{(-q;q)_j}{(q;q)_j} \mathrm{e}^{-q^j t} z^j\right] \times \left[\sum_{m=0}^{\infty} \frac{(-1;q)_m}{(q;q)_m} z^m\right]. \end{aligned}$$

However, according to the Heine formula for basic hypergeometric functions (Gasper & Rahman, 1990),

$$\sum_{m=0}^{\infty} \frac{(-1;q)_m}{(q;q)_m} z^m = {}_1\Phi_0\left[\begin{matrix}-1; \\ \text{—};\end{matrix}\; q; z\right] = \frac{(-z;q)_\infty}{(z;q)_\infty},$$

therefore

$$G(t,z) = \frac{(-z;q)_\infty}{(z;q)_\infty} \sum_{j=0}^{\infty}(-1)^j \frac{(-q;q)_j}{(q;q)_j} \mathrm{e}^{-q^j t} z^j,$$

and this, according to (Iserles, 1993), is the Dirichlet series expansion of the solution of the *pantograph equation* (2.4).

To evaluate the initial condition we again sum a basic hypergeometric series with the Heine formula,

$$\sum_{j=0}^{\infty}(-1)^j \frac{(-q;q)_j}{(q;q)_j} z^j = {}_1\Phi_0 \left[\begin{matrix} -q; \\ —; \end{matrix} \; q; z \right] = \frac{(qz;q)_\infty}{(-z;q)_\infty}.$$

Therefore

$$G(0,z) = \frac{(-z;q)_\infty}{(z;q)_\infty} \times \frac{(qz;q)_\infty}{(-z;q)_\infty} = \frac{1}{1-z},$$

affirming (2.5). □

Corollary 10 *The Taylor expansion (in t) of the function G is*

$$G(t,z) = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} \frac{(-z;q)_k}{(z;q)_{k+1}} t^k. \tag{2.6}$$

Proof According to (Iserles, 1993), the solution of the pantograph equation

$$y'(t) = ay(t) + by(qt) + cy'(qt), \quad t \geq 0, \qquad y(0) = y_0, \tag{2.7}$$

where $a, b, c \in \mathbb{C}$, $a \neq 0$, $|c| < 1$, can be expanded into the Taylor series

$$y(t) = y_0 \sum_{k=0}^{\infty} \frac{1}{k!} \frac{(-b/a;q)_k}{(c;q)_k} (at)^k.$$

Letting $a = -1$, $b = -z$, $c = qz$ and $y_0 = 1/(1-z)$ yields (2.6). □

We note as an aside that, according to (Iserles, 1993) and (Iserles & Liu, 1994), the solution of (2.7) exists and it is unique subject to the inequality $|c| < 1$. Moreover, $\mathrm{Re}\, a < 0$, $|a| > |b|$ implies that the solution is asymptotically stable. Thus, thanks to the restriction $|z| < 1$, we deduce that $\lim_{t\to\infty} G(t,z) = 0$.

Next we consider recurrence relations that are obeyed by the sequence $\{R_n\}_{n=0}^{\infty}$. A differential recurrence follows by substitution in (1.11), namely

$$R'_{n-1} - R'_n = q^{n-1} R_{n-1} + q^n R_n, \qquad n = 1, 2, \ldots. \tag{2.8}$$

This recurrence can be rewritten in the form (1.12).

Several other relations can be derived by a moderate effort. For example, we can express R_n in terms of R_{n-1} and its derivatives at t and qt (a differential recurrence with rescaling). Thus, let

$$\varphi_n(t) := (1-q^n)R_n(t) + R_{n-1}(qt) - q^{n-1}R_{n-1}(t) + R'_{n-1}(t) - qR'_{n-1}(qt), \quad n = 1, 2, \ldots. \tag{2.9}$$

It is a trite exercise to verify that

$$\begin{aligned} \langle \mathrm{e}^{-q^\ell t}, R_{n-1}(qt) \rangle &= q^{-1} \langle \mathrm{e}^{-q^{\ell-1} t}, R_{n-1} \rangle, \\ \langle \mathrm{e}^{-q^\ell t}, R'_{n-1}(qt) \rangle &= -q^{-1} + q^{\ell-2} \langle \mathrm{e}^{-q^{\ell-1} t}, R_{n-1} \rangle, \end{aligned} \qquad \ell = 1, 2, \ldots,$$

consequently

$$\begin{aligned}\langle \mathrm{e}^{-q^\ell t}, \varphi_n\rangle &= (1-q^n)\langle \mathrm{e}^{-q^\ell t}, R_n\rangle + q^{-1}\langle \mathrm{e}^{-q^{\ell-1} t}, R_{n-1}\rangle - q^{n-1}\langle \mathrm{e}^{-q^\ell t}, R_{n-1}\rangle \\ &\quad + q^\ell\langle \mathrm{e}^{-q^\ell t}, R_{n-1}\rangle - q^{\ell-1}\langle \mathrm{e}^{-q^{\ell-1} t}, R_{n-1}\rangle, \qquad \ell = 1,2,\ldots.\end{aligned}$$

Recalling that each R_m is orthogonal to $\exp(-q^\ell t)$ for $\ell = 0,1,\ldots,m-1$, we thus deduce the relations

$$\langle \mathrm{e}^{-q^\ell t}, \varphi_n\rangle = 0, \qquad \ell = 0,1,\ldots,n-1.$$

Therefore φ_n is a scalar multiple of R_n. But Corollary 4 implies $R_n(0) = 1$, whilst (2.8) yields $R_n'(0) = -(2-q^n-q^{n+1})/(1-q)$. Substitution in (2.9) verifies at once $\varphi_n(0) = 0$, thus leading to the conclusion that $\varphi_n \equiv 0$.

Proposition 11 *The sequence $\{R_n\}_{n=0,1,\ldots}$ satisfies the differential recurrence relation with rescaling*

$$(1-q^n)R_n(t) = q^{n-1}R_{n-1}(t) - R_{n-1}(qt) - R_{n-1}'(t) + qR_{n-1}'(qt), \qquad n = 1,2,\ldots.$$

□

Finally, we report a pure recurrence relation with rescaling.

Proposition 12 *For every $n = 2,3,\ldots$ we have the relation*

$$(1-q^n)R_n(t) = (1+q^{n-1})R_{n-1}(t) - (1+q^n)R_{n-1}(qt) + (1-q^{n-1})R_{n-2}(qt). \quad (2.10)$$

Proof Although (2.10) can be proved by comparing coefficients in (2.3), it is perhaps more interesting to use the generating function G. Comparing the coefficients in (2.6) affirms the identity

$$(1-z)[G(t,z) + zG(qt,z)] = (1+z)[G(t,gz) - qzG(qt,qz)],$$

which we rewrite in the form

$$\begin{aligned}G(t,z) - G(t,gz) &= z[G(t,z) + G(t,qz)] - z[G(qt,z) - qG(qt,gz)] \\ &\quad + z^2[G(qt,z) - qG(qt,qz)].\end{aligned}$$

The recurrence (2.10) follows at once, by substituting the definition of the generating function G. □

3. Convergence of projections

Let $\lambda_0, \lambda_1, \ldots$ be given positive numbers. We consider the approximation of a function $f \in L^2[0,\infty)$ by projections onto each H_n. We henceforth consider only the inner product $\langle f,g\rangle = \int_0^\infty f(t)g(t)\,\mathrm{d}t$, although our discussion can be generalized with minimal effort to the inner product (2.1).

The orthogonal projection of f onto H_n can be written as a generalized Fourier series, which in turn can be expressed explicitly in terms of the Fourier transform of f, so that

$$F_n(t) = \sum_{m=0}^{n} \frac{\langle f, r_m\rangle}{\langle r_m, r_m\rangle} r_m(t) = 2\sum_{m=0}^{n-1} \lambda_m \langle f, r_m\rangle r_m(t), \qquad n = 0, 1, \ldots, \tag{3.1}$$

where (cf. (1.6))

$$r_m(t) = \sum_{\ell=0}^{n} r_{m,\ell} \exp(-\lambda_\ell t),$$

$$\text{implies} \qquad \langle f, r_m\rangle = \sum_{\ell=0}^{m} r_{m,\ell} \int_0^\infty f(t)\mathrm{e}^{-\lambda_\ell t}\,\mathrm{d}t = \sum_{\ell=0}^{m} r_{m,\ell}\hat{f}(-\mathrm{i}\lambda_\ell)., \tag{3.2}$$

The behaviour of the Fourier transform is entwined with the convergence of the projections. Let us use $\mathcal{H}$ to denote the closure of the subspace $H_1 + H_2 + \cdots$ in the L^2 norm.

Proposition 13 *Let $f \in \mathcal{H}$. If the function*

$$\Lambda(z) := \sum_{n=0}^{\infty} \lambda_n z^n$$

is analytic in an open disc centred at the origin and we have the inequality

$$|\langle f, r_n\rangle| \le \omega^n |\langle f, r_0\rangle|, \qquad n = 0, 1, \ldots, \tag{3.3}$$

where $\omega \in (0,1)$ is a point in the disc, then

$$||f - F_n||^2 \le 2|\langle f, r_0\rangle|^2 \sum_{m=n}^{\infty} \lambda_m \omega^{2m}, \tag{3.4}$$

and the right hand side converges to zero as n tends to infinity.

Proof It follows from (3.1) by the Parseval equality that

$$||f - F_n||^2 = 2\sum_{m=n}^{\infty} \lambda_m |\langle f, r_m\rangle|^2,$$

and we deduce (3.4) from (3.3) and the analyticity of Λ at ω^2. □

Let us assume further that $\hat{f}$ is analytic in the closed unit disc. Substituting

$$\hat{f}(\mathrm{i}z) = \sum_{k=0}^{\infty} \frac{\varphi_k}{k!} z^k$$

into (3.2) yields

$$\langle f, r_m \rangle = \sum_{k=0}^{\infty} \frac{\varphi_k}{k!} \sum_{\ell=0}^{m} r_{m,\ell} (-\lambda_\ell)^k = \sum_{k=0}^{\infty} \frac{\varphi_k}{k!} r_m^{(k)}(0), \qquad m = 0, 1, \ldots. \tag{3.5}$$

This formula and inequality (3.3) motivate our interest in the magnitude of the derivatives of r_m at the origin.

We restrict our attention in the remainder of this section to the parameters $\lambda_k = q^k$, $k = 0, 1, \ldots$, that have already featured in Section 2. According to (2.6) we have

$$\sum_{n=0}^{\infty} R_n^{(k)}(0) z^n = \frac{\partial^k}{\partial t^k} G(0, z) = (-1)^k \frac{(-z; q)_k}{(z; q)_{k+1}}, \qquad k = 0, 1, \ldots,$$

hence the recursion

$$\begin{aligned} \sum_{n=0}^{\infty} R_n^{(k)}(0) z^n &= -\frac{1 + q^{k-1} z}{1 - q^k z} \sum_{n=0}^{\infty} R_n^{(k-1)}(0) z^n \\ &= -\sum_{n=0}^{\infty} \left[\sum_{\ell=0}^{n} R_{n-\ell}^{(k-1)}(0) q^{k\ell} \right] z^n - \sum_{n=1}^{\infty} \left[\sum_{\ell=1}^{n} R_{n-\ell}^{(k-1)}(0) q^{k\ell-1} \right] z^n. \end{aligned}$$

We deduce that

$$R_n^{(k)}(0) = -\sum_{\ell=0}^{n} R_{n-\ell}^{(k-1)}(0) q^{k\ell} - q^{k-1} \sum_{\ell=0}^{n-1} R_{n-1-\ell}^{(k-1)}(0) q^{k\ell}, \qquad n = 0, 1, \ldots, \ k = 1, 2, \ldots. \tag{3.6}$$

Proposition 14 *The derivatives of R_n at the origin obey the inequality*

$$|R_n^{(k)}(0)| \le \frac{(-1; q)_k}{(q; q)_k}, \qquad n, k = 0, 1, \ldots. \tag{3.7}$$

Proof We use induction on the derivative order k. The inequality is true for $k = 0$ because, by Corollary 4, $R_n(0) = 1$. We thus assume its correctness for $k - 1$ and employ (3.6) to argue that

$$\begin{aligned} |R_n^{(k)}(0)| &\le \frac{(-1; q)_{k-1}}{(q; q)_{k-1}} \left[\sum_{\ell=0}^{n} q^{k\ell} + q^{k-1} \sum_{\ell=0}^{n-1} q^{k\ell} \right] \\ &\le \frac{(-1; q)_{k-1}}{(q; q)_{k-1}} \left[\frac{1}{1 - q^k} + \frac{q^{k-1}}{1 - q^k} \right] = \frac{(-1; q)_k}{(q; q)_k} \end{aligned}$$

which completes the proof. □

Corollary 15

$$|R_n^{(k)}(0)| \le \frac{(-1; q)_\infty}{(q; q)_\infty}, \qquad n, k = 0, 1, \ldots. \tag{3.8}$$

Proof This is immediate from (3.7), since the sequence

$$\left\{\frac{(-1;q)_{k-1}}{(q;q)_{k-1}} : k=0,1,\ldots\right\}$$

is increasing for every $q \in (0,1)$. □

Of course, analyticity of $\hat{f}$ in the closed unit disc implies absolute convergence of its Taylor series at $z=1$. Setting

$$\sigma := \sum_{k=0}^{\infty} \frac{|\varphi_k|}{k!} < \infty,$$

we thereby deduce from (3.8) that

$$|\langle f, R_n\rangle \le \sigma \frac{(-1;q)_\infty}{(q;q)_\infty}, \qquad n=0,1,\ldots.$$

Consequently, (3.5) and (3.8) imply the bound

$$|\langle f, R_n\rangle| \le \sigma \frac{(-1;q)_\infty}{(q;q)_\infty}, \qquad n=0,1,\ldots,$$

and, when $f \in \mathcal{H}$, the Parseval theorem yields the expression

$$\|f - F_N\|^2 = 2\sum_{n=N}^{\infty} q^n |\langle f, R_n\rangle|^2.$$

Therefore, our bound on $|\langle f, R_n\rangle|$ establishes the following result.

Theorem 16 *If $f \in \mathcal{H}$ and $\hat{f}$ is analytic in the closed unit disc, then*

$$\|f - F_n\| \le \sigma^* q^{N/2} \stackrel{N\to\infty}{\longrightarrow} 0, \tag{3.9}$$

where

$$\sigma^* = \sigma\sqrt{\frac{2}{1-q}} \times \frac{(-1;q)_\infty}{(q;q)_\infty}.$$

□

Furthermore, since

$$\sum_{k=0}^{\infty} \frac{|\varphi_k|}{k!} \le \int_0^\infty \mathrm{e}^t |f(t)|\,\mathrm{d}t. \tag{3.10}$$

Therefore boundedness of this integral implies analyticity of $\hat{f}$ in the closed unit disc.

Let us return to the interesting special case $f(t) = \exp(-\lambda t)$. By (1.2),

$$\langle \mathrm{e}^{-\lambda t}, R_n\rangle = \langle g_\lambda, R_n\rangle = \hat{R}_n(-\mathrm{i}\lambda)$$

and (1.5) yields

$$\langle \mathrm{e}^{-\lambda t}, r_n \rangle = \frac{\prod_{k=0}^{n-1}(\lambda - \lambda_k)}{\prod_{k=0}^{n}(\lambda + \lambda_k)}, \qquad n = 0, 1, \ldots.$$

Specialising to $\lambda_\ell = q^\ell$, we obtain

$$\langle \mathrm{e}^{-\lambda t}, R_n \rangle = \frac{1}{\lambda} \frac{(1/\lambda; q)_n}{(-1/\lambda; q)_{n+1}}, \qquad n = 0, 1, \ldots. \tag{3.11}$$

We thus deduce by the method of proof of Theorem 16 that

$$F_n(t) = \frac{2}{\lambda} \sum_{n=0}^{N} \frac{(1/\lambda; q)_n}{(-1/\lambda; q)_{n+1}} q^n R_n(t) \tag{3.12}$$

converges in norm to the orthogonal projection of $\exp(-\lambda t)$ on $\mathcal{H}$.

The projection of $f(t) = \exp(-\lambda t)$, $\lambda > 1$, onto H_n is

$$F_N(t) = \frac{2}{\lambda} \sum_{n=0}^{N} \frac{(1/\lambda; q)_n}{(-1/\lambda; q)_{n+1}} q^n R_n(t).$$

Letting $N \to \infty$, it is easy to verify that

$$F_N(t) \to F(t) = \frac{2}{1+\lambda} \sum_{n=0}^{\infty} \frac{(1/\lambda; q)_n}{(-q/\lambda; q)_n} q^n R_n(t). \tag{3.13}$$

We next substitute the explicit expression for R_n from (2.3), whence exchanging the order of summation yields

$$\begin{aligned} F(t) &= \frac{2}{1+\lambda} \sum_{m=0}^{\infty} (-1)^m \frac{(-q; q)_m}{(q; q)_m} \mathrm{e}^{-q^m t} \sum_{n=m}^{\infty} \frac{(1/\lambda; q)_n (-1; q)_{n-m}}{(-q/\lambda; q)_n (q; q)_{n-m}} q^n \\ &= \frac{2}{1+\lambda} \sum_{m=0}^{\infty} (-1)^m \frac{(-q; q)_m}{(q; q)_m} q^m \mathrm{e}^{-q^m t} \sum_{n=0}^{\infty} \frac{(1/\lambda; q)_{m+n} (-1; q)_n}{(q; q)_n (-q/\lambda; q)_{m+n}} q^n \\ &= \frac{2}{1+\lambda} \sum_{m=0}^{\infty} (-1)^m \frac{(-q; q)_m (1/\lambda; q)_m}{(q; q)_m (-q/\lambda; q)_m} q^m \mathrm{e}^{-q^m t} \sum_{n=0}^{\infty} \frac{(q^m/\lambda; q)_n (-1; q)_n}{(q; q)_n (-q^{m+1}/\lambda; q)_n} q^n \\ &= \frac{2}{1+\lambda} \sum_{m=0}^{\infty} (-1)^m \frac{(-q; q)_m (1/\lambda; q)_m}{(q; q)_m (-q/\lambda; q)_m} q^m \mathrm{e}^{-q^m t} {}_2\Phi_1 \left[\begin{matrix} q^m/\lambda, -1; \\ -q^{m+1}/\lambda; \end{matrix} \; q, q \right] \end{aligned} \tag{3.14}$$

– we refer to (Gasper & Rahman, 1990) for the terminology of basic hypergeometric series.

The ${}_2\Phi_1$ series in (3.14) can be summed with the q-Gauß formula (Gasper & Rahman, 1990, formula II.8, page 236),

$${}_2\Phi_1 \left[\begin{matrix} q^m/\lambda, -1; \\ -q^{m+1}/\lambda; \end{matrix} \; q, q \right] = \frac{(-q; q)_\infty (q^{m+1}/\lambda; q)_\infty}{(q; q)_\infty (-q^{m+1}/\lambda; q)_\infty}.$$

Substitution in (3.14) and elementary simplification result in

$$F(t) = \frac{1}{\lambda} \frac{(-1;q)_\infty (1/\lambda;q)_\infty}{(q;q)_\infty (-1/\lambda;q)_\infty} \sum_{m=0}^{\infty} (-1)^m \frac{(-q;q)_m}{(q;q)_m} \frac{q^m}{1-q^m/\lambda} \mathrm{e}^{-q^m t}. \tag{3.15}$$

Note that (3.15) is valid for all $\lambda > 0$, $\lambda \neq q^k$ for $k \geq 0$. In the case $\lambda = q^k$ it is easy to prove that $F(t) = \exp(-q^k t)$.

Finally, we explicitly evaluate $F(0)$ from (3.15). In the course of our analysis we twice use an explicit formula for the summation of ${}_1\Phi_0$ series (Gasper & Rahman, 1990, formula II.3, page 236).

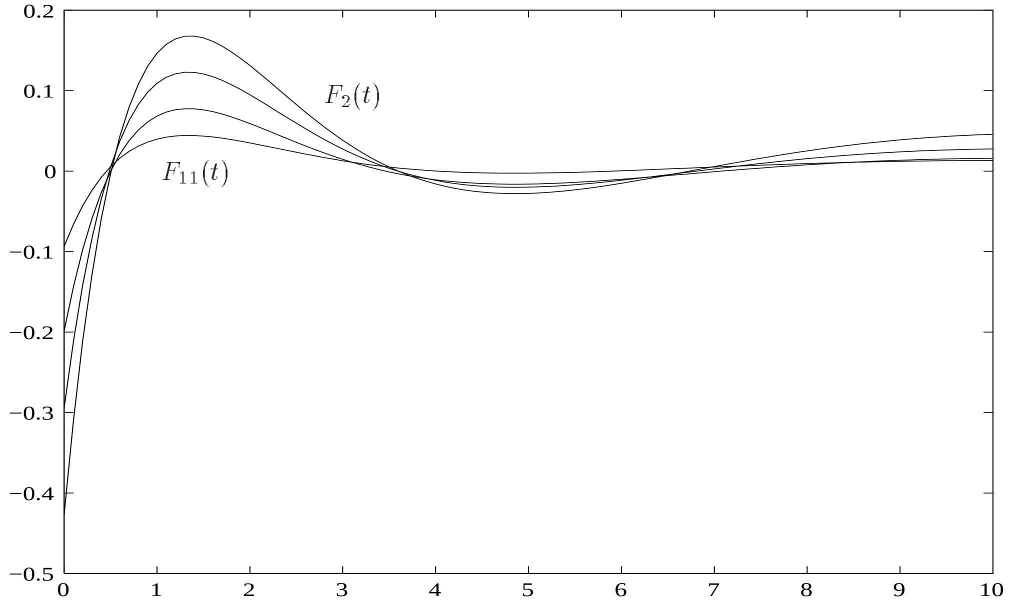

Figure 1 The functions F_N for $N = 2, 5, 8, 11$, $q = \frac{1}{2}$ and $\lambda = \frac{3}{2}$.

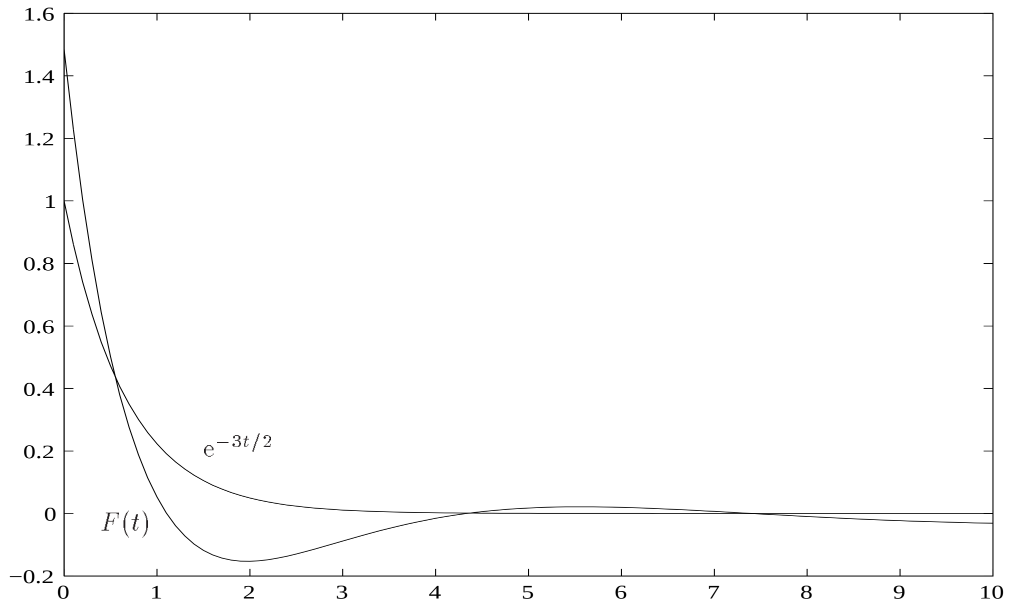

Figure 2 The functions F and $\exp(-\lambda t)$ for $q = \frac{1}{2}$ and $\lambda = \frac{3}{2}$.

Bearing in mind that $0 < 1/\lambda < 1$, we expand into series,

$$\begin{aligned}
&\sum_{m=0}^{\infty}(-1)^m\frac{(-q;q)_m}{(q;q)_m}\frac{q^m}{1-q^m/\lambda}\\
&= \sum_{m=0}^{\infty}(-1)^m\frac{(-q;q)_m}{(q;q)_m}q^m\sum_{\ell=0}^{\infty}\frac{q^{m\ell}}{\lambda^\ell}\\
&= \sum_{\ell=0}^{\infty}\frac{1}{\lambda^\ell}{}_1\Phi_0\left[\ -q;—;\ q,-q^{\ell+1}\right]\\
&= \sum_{\ell=0}^{\infty}\frac{1}{\lambda^\ell}\frac{(q^{\ell+2};q)_\infty}{(-q^{\ell+1};q)_\infty} = \lambda\left[\sum_{\ell=0}^{\infty}\frac{(q^{\ell+1};q)_\infty}{(-q^\ell;q)_\infty}\frac{1}{\lambda^\ell}-\frac{(q;q)_\infty}{(-1;q)_\infty}\right]\\
&= \lambda\frac{(q;q)_\infty}{(-1;q)_\infty}\left[\sum_{\ell=0}^{\infty}\frac{(-1;q)_\ell}{(q;q)_\ell}\lambda^{-\ell}-1\right]\\
&= \lambda\frac{(q;q)_\infty}{(-1;q)_\infty}\left\{{}_1\Phi_0\left[\begin{matrix}-1;\\ —;\end{matrix}\ q,\lambda^{-1}\right]-1\right\} = \frac{(q;q)_\infty}{(-1;q)_\infty}\left[\frac{(-1/\lambda;q)_\infty}{(1/\lambda;q)_\infty}-1\right].
\end{aligned}$$

Therefore, substitution in (3.15) proves that

$$F(0) = 1-\frac{(1/\lambda;q)_\infty}{(-1/\lambda;q)_\infty}.$$

Note that, unless $\lambda = q^{-m}$ for a nonnegative integer m, it follows that $F(0) \neq 1$. This constitutes a formal proof of the statement that the sequence $\{F_N\}$ does not converge to $\exp(-\lambda t)$; of course we have already provided a stronger result in Proposition 7. We can also use the infinite product 1.9 to characterize the lmit of the projections $\{F_N\}$. Indeed, the proof of Proposition 7 implies that

$$\lim_{N\to\infty}\hat{f}(z)-\hat{F}_N(z) = \frac{-\mathrm{i}}{z-\mathrm{i}\lambda}\Delta(\mathrm{i}z)\Delta(\lambda). \tag{3.16}$$

In Figure 1 we display the functions F_N for different values of N in the case $q = \frac{1}{2}$, $\lambda = \frac{3}{2}$. It illustrates vividly our observation that $F_N \to F$ yet, as can be seen in Figure 2, the function F is distinct from $\exp(-\frac{3}{2}t)$.

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