Norm estimates for the ℓ^2-inverses of multivariate Toeplitz matrices

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This paper bounds the ℓ^2-norms of inverses of certain Toeplitz matrices arising from Pólya frequency functions.

1. Introduction

Let $\varphi: \mathcal{R}^d \to \mathcal{R}$ be an even continuous function of at most polynomial growth. Associated with this function is a symmetric bi-infinite multivariate Toeplitz matrix

$$\Phi = (\varphi(j-k))_{j,k\in\mathcal{Z}^d}\,. \tag{1.1}$$

Every finite subset I of $\mathcal{Z}^d$ determines a finite submatrix of Φ given by

$$\Phi_I := (\varphi(j-k))_{j,k\in I}\,. \tag{1.2}$$

We are interested in upper bounds on the ℓ^2-norm of the inverse matrix Φ^{-1}, that is the quantity

$$\|\Phi_I^{-1}\| = 1\Big/\min\{\|x\|_2 : \|\Phi_I x\|_2 = 1, \quad x \in \mathcal{R}^I\}, \tag{1.3}$$

where $\|x\|_2^2 = \sum_{j\in I} x_j^2$ for $x = (x_j)_{j\in I}$. The type of bound we seek follows the pattern of results in Baxter (1991). Specifically, we let $\hat\varphi$ be the distributional Fourier transform of φ in the sense of Schwartz (1966), which we assume to be a measurable function on $\mathcal{R}^d$. We let $e := (1,\ldots,1)^T \in \mathcal{R}^d$ and set

$$\tau_{\hat\varphi} := \sum_{j\in\mathcal{Z}^d} |\hat\varphi(\pi e + 2\pi j)| \tag{1.4}$$

whenever the right hand side of this equation is meaningful. Then, for a certain class of *radially symmetric* functions, Baxter proved that

$$\|\Phi_I^{-1}\| \le 1/\tau_{\hat\varphi} \tag{1.5}$$

for every finite subset I of $\mathcal{Z}^d$. In this paper we extend this bound to a wider class of functions which need not be radially symmetric. For instance, we show that (1.5) holds for the class of functions

$$\varphi(x) = (\|x\|_1 + c)^\gamma, \qquad x \in \mathcal{R}^d,$$

where $\|x\|_1 = \sum_{j=1}^d |x_j|$ is the ℓ_1-norm of x, and either $\gamma < 1$, $c > 0$ or $0 < \gamma < 1$ and $c = 0$.

Our analysis develops the methods used by Baxter (1991). However, here we emphasize the importance of certain properties of Pólya frequency functions and Pólya frequency sequences (due to I. J. Schoenberg) in order to obtain estimates like (1.5).

In Section 2 we discuss Fourier transform techniques which we need to prove our bound. Further, the results of this section improve on the treatment of Baxter (1991), in that the condition of admissibility (see Definition 3.2 of Baxter (1991)) is shown to be unnecessary. Section 3 contains a discussion of the class of functions φ for which we will prove the bound (1.4). The final section contains the proof of our main result.

2. Preliminary facts

We begin with a rather general framework. Let $\varphi\colon \mathcal{R}^d \to \mathcal{R}$ be a continuous function of polynomial growth. Thus φ possesses a Fourier transform in the sense of Schwartz (1966) which we shall assume is almost everywhere equal to a Lebesgue measurable function on $\mathcal{R}^d$. Given a nonzero real sequence $(y_j)_{j\in\mathcal{Z}^d}$ of finite support and points $(x^j)_{j\in\mathcal{Z}^d}$ in $\mathcal{R}^d$, we introduce the function $F\colon \mathcal{R}^d \to \mathcal{R}$ given by

$$F(x) = \sum_{j,k\in\mathcal{Z}^d} y_j y_k \varphi(x + x^j - x^k), \qquad x \in \mathcal{R}^d. \tag{2.1}$$

Thus the value of F at zero is given by

$$F(0) = \sum_{j,k\in\mathcal{Z}^d} y_j y_k \varphi(x^j - x^k), \tag{2.2}$$

which is the quadratic form whose study is the object of this paper. We observe that the Fourier transform of F is the tempered distribution

$$\hat{F}(\xi) = \Bigl|\sum_{j\in\mathcal{Z}^d} y_j e^{ix^j\xi}\Bigr|^2 \hat{\varphi}(\xi), \qquad \xi \in \mathcal{R}^d. \tag{2.3}$$

Further, if $\hat{F}$ is an absolutely integrable function, then the inversion formula provides the equation

$$F(0) = (2\pi)^{-d} \int_{\mathcal{R}^d} \hat{F}(\xi)\, d\xi, \tag{2.4}$$

since F is the inverse distributional Fourier transform of $\hat{F}$ and this coincides with the classical inverse transform when $\hat{F} \in L^1(\mathcal{R}^d)$. In other words, we have the equation

$$\sum_{j,k\in\mathcal{Z}^d} y_j y_k \varphi(x^j - x^k) = (2\pi)^{-d} \int_{\mathcal{R}^d} \Bigl|\sum_{j\in\mathcal{Z}^d} y_j e^{ix^j\xi}\Bigr|^2 \hat{\varphi}(\xi)\, d\xi \tag{2.5}$$

when $\hat{F}$ is absolutely integrable. If we make the further assumption that $\hat{\varphi}$ is *one-signed* almost everywhere on $\mathcal{R}^d$, and the points $(x^j)_{j\in\mathcal{Z}^d}$ form a subset of the integers $\mathcal{Z}^d$, then it is possible to improve (2.5). First observe that dissecting $\mathcal{R}^d$ into integer translates of the cube $[0, 2\pi]^d$ provides the relations

$$\begin{aligned}
\sum_{j,k\in\mathcal{Z}^d} y_j y_k \varphi(j - k) &= (2\pi)^{-d} \int_{\mathcal{R}^d} \Bigl|\sum_{j\in\mathcal{Z}^d} y_j e^{ij\xi}\Bigr|^2 \hat{\varphi}(\xi)\, d\xi \\
&= \sum_{k\in\mathcal{Z}^d} (2\pi)^{-d} \int_{[0,2\pi]^d} \Bigl|\sum_{j\in\mathcal{Z}^d} y_j e^{ij\xi}\Bigr|^2 \hat{\varphi}(\xi + 2\pi k)\, d\xi \\
&= (2\pi)^{-d} \int_{[0,2\pi]^d} \Bigl|\sum_{j\in\mathcal{Z}^d} y_j e^{ij\xi}\Bigr|^2 \sigma(\xi)\, d\xi,
\end{aligned} \tag{2.6}$$

where

$$\sigma(\xi) = \sum_{k \in \mathcal{Z}^d} \hat{\varphi}(\xi + 2\pi k), \qquad \text{for almost every } \xi \in \mathcal{R}^d \tag{2.7}$$

and the monotone convergence theorem justifies the exchange of summation and integration. Further, we see that another consequence of the condition that $\hat{\varphi}$ be one-signed is the bound

$$\Big| \sum_{j \in \mathcal{Z}^d} y_j e^{ij\xi} \Big|^2 |\hat{\varphi}(\xi)| < \infty$$

for almost every point $\xi \in [0, 2\pi]^d$, because the left hand side of (2.6) is *a fortiori* finite. This implies that σ is almost everywhere finite, since the set of all zeros of a nonzero trigonometric polynomial has measure zero. This is well known, but we include its short proof for completeness below. Following Rudin (1986), we shall say that a continuous function $f\colon \mathcal{C}^d \to \mathcal{C}$ is an *entire function of d complex variables* if, for every point $(w_1, \ldots, w_d) \in \mathcal{C}^d$ and for every $j \in \{1, \ldots, d\}$, the mapping

$$\mathcal{C} \ni x \mapsto f(w_1, \ldots, w_{j-1}, z, w_{j+1}, \ldots, w_d)$$

is an entire function of one complex variable.

Lemma 2.1. *Given complex numbers* $(a_j)_{j=1}^n$ *and a set of distinct points* $(x^j)_{j=1}^n$ *in* $\mathcal{R}^d$*, we let* $p\colon \mathcal{R}^d \to \mathcal{C}$ *be the function*

$$p(\xi) = \sum_{j=1}^n a_j e^{ix^j \xi}, \qquad \xi \in \mathcal{R}^d.$$

Then p enjoys the following properties:

(i) p is identically zero if and only if $a_j = 0$, $1 \le j \le n$.

(ii) p is nonzero almost everywhere unless $a_j = 0$, $1 \le j \le n$.

Proof.

(i) Suppose p is identically zero. Choose any $j \in \{1, \ldots, n\}$ and let $f_j\colon \mathcal{R}^d \to \mathcal{R}$ be a continuous function of compact support such that $f_j(x^k) = \delta_{jk}$ for $1 \le k \le n$. Then

$$a_j = \sum_{k=1}^n a_k f_j(x^k) = (2\pi)^{-d} \int_{\mathcal{R}^d} \sum_{k=1}^n a_k e^{ix^k \xi} \hat{f}_j(\xi)\, d\xi = 0.$$

The converse is obvious.

(ii) Let $f\colon \mathcal{C}^d \to \mathcal{C}$ be an entire function and let

$$Z = \{x \in \mathcal{R}^d : f(x) = 0\}.$$

If $\mathrm{vol}_d Z$ is a set of positive Lebesgue measure in $\mathcal{R}^d$, then we shall prove that f is identically zero, which implies the required result. We proceed by induction on the dimension d. When $d = 1$, we see that f is an entire function of one complex variable with uncountably many zeros, because Z is a set of real numbers with positive Lebesgue measure. Such a function must vanish everywhere. Therefore suppose that the result is true for $d-1$ for some $d \geq 2$. Fubini's theorem provides the relation

$$0 < \mathrm{vol}_d Z = \int_{\mathcal{R}^{d-1}} \mathrm{vol}_1 Z(x_2, \ldots, x_d)\, dx_2 \ldots dx_d,$$

where

$$Z(x_2, \ldots, x_d) = \{x_1 \in \mathcal{R} : (x_1, \ldots, x_d) \in Z\}.$$

Thus there is a set, X say, in $\mathcal{R}^{d-1}$ of positive $(d-1)$-dimensional Lebesgue measure such that $\mathrm{vol}_1 Z(x_2, \ldots, x_d)$ is positive for every $(x_2, \ldots, x_d) \in X$, and therefore the entire function $\mathcal{C} \ni z \mapsto f(z, x_2, \ldots, x_d)$ vanishes for all $z \in \mathcal{C}$, because $Z(x_2, \ldots, x_d)$ is an uncountable set. Thus, choosing any $z_1 \in \mathcal{C}$, we see that the entire function of $d-1$ complex variables defined by

$$(z_2, \ldots, z_d) \mapsto f(z_1, z_2, \ldots, z_d), \qquad (z_2, \ldots, z_d) \in \mathcal{C}^{d-1},$$

vanishes for all $(z_2, \ldots, z_d)$ in X, which is a set of positive $(d-1)$- dimensional Lebesgue measure. By induction hypothesis, we deduce that

$$f(z_1, z_2, \ldots, z_d) = 0 \text{ for all } z_2, \ldots, z_d \in \mathcal{C},$$

and since z_1 can be any complex number, we conclude that f is identically zero. ■

We can now derive our first bounds on the quadratic form (2.2). For any measurable function $g\colon [0, 2\pi]^d \to \mathcal{R}$, we recall the definitions of the essential supremum

$$\text{ess sup } g = \inf\{c \in \mathcal{R} : g(x) \leq c \text{ for almost every } x \in [0, 2\pi]^d\} \tag{2.8}$$

and the essential infimum

$$\text{ess inf } g = \sup\{c \in \mathcal{R} : g(x) \geq c \text{ for almost every } x \in [0, 2\pi]^d\}. \tag{2.9}$$

Thus (2.6) and the Parseval relation provide the inequalities

$$\text{ess inf } \sigma \sum_{j \in \mathcal{Z}^d} y_j^2 \leq \sum_{j,k \in \mathcal{Z}^d} y_j y_k \varphi(x^j - x^k) \leq \text{ess sup } \sigma \sum_{j \in \mathcal{Z}^d} y_j^2. \tag{2.10}$$

Let V be the vector space of real sequences $(y_j)_{j \in \mathcal{Z}^d}$ of finite support for which the function $\hat{F}$ of (2.3) is absolutely integrable. We have seen that, when $\hat{\varphi}$ is one-signed, (2.10) is valid for every element $(y_j)_{j \in \mathcal{Z}^d}$ of V. Of course, at this stage there is no guarantee that $V \neq \{0\}$ or that the bounds are finite. Nevertheless, we identify below a case when the bounds (2.10) cannot be improved. This will be of relevance later.

Proposition 2.2. *Let P be a nonzero trigonometric polynomial such that the principal ideal $\mathcal{I}$ generated by P, that is the set*

$$\mathcal{I} = \{P \cdot T : T \text{ a trigonometric polynomial}\}, \tag{2.11}$$

consists of trigonometric polynomials whose Fourier coefficient sequences are elements of V. Further, suppose that there is a point η at which σ is continuous and $P(\eta) \neq 0$. Then we can find a sequence $\{(y_j^{(n)})_{j\in\mathcal{Z}^d} : n = 1, 2, \ldots\}$ in V such that

$$\lim_{n\to\infty} \sum_{j,k\in\mathcal{Z}^d} y_j^{(n)} y_k^{(n)} \varphi(j-k) \Big/ \sum_{j\in\mathcal{Z}^d} [y_j^{(n)}]^2 = \sigma(\eta). \tag{2.12}$$

Proof. We follow Baxter (1991) and introduce the nth degree tensor product Fejér kernel

$$K_n(\xi) := \prod_{j=1}^{d} \frac{\sin^2 n\xi_j/2}{n \sin^2 \xi_j/2} = \Big| n^{-d/2} \sum_{\substack{k\in\mathcal{Z}^d \\ 0\le k< en}} e^{ik\xi} \Big|^2 =: |L_n(\xi)|^2, \qquad \xi \in \mathcal{R}^d, \tag{2.13}$$

where $e = (1, \ldots, 1)^T \in \mathcal{R}^d$. Then the function $P(\cdot)L_n(\cdot - \eta)$ is a member of $\mathcal{I}$ and we choose $(y_j^{(n)})_{j\in\mathcal{Z}^d}$ to be its Fourier coefficient sequence. The Parseval relation provides the equation

$$\sum_{j\in\mathcal{Z}^d} [y_j^{(n)}]^2 = (2\pi)^{-d} \int_{[0,2\pi]^d} \Big| P^2(\xi) \Big| K_n(\xi - \eta)\, d\xi \tag{2.14}$$

and the approximate identity property of the Fejér kernel (Zygmund (1988), p.86) implies that

$$\begin{aligned} P^2(\eta) &= \lim_{n\to\infty} (2\pi)^{-d} \int_{[0,2\pi]^d} \Big| P^2(\xi) \Big| K_n(\xi - \eta)\, d\xi \\ &= \lim_{n\to\infty} \sum_{j\in\mathcal{Z}^d} [y_j^{(n)}]^2. \end{aligned} \tag{2.15}$$

Further, because σ is continuous at η, we also have the relations

$$\begin{aligned} P^2(\eta)\sigma(\eta) &= \lim_{n\to\infty} (2\pi)^{-d} \int_{[0,2\pi]^d} P^2(\xi) K_n(\xi - \eta) \sigma(\xi)\, d\xi \\ &= \lim_{n\to\infty} \sum_{j,k\in\mathcal{Z}^d} y_j^{(n)} y_k^{(n)} \varphi(j-k). \end{aligned} \tag{2.16}$$

Hence (2.15) and (2.16) provide equation (2.13). ∎

Corollary 2.3. *If σ attains its essential infimum (resp. supremum) at a point of continuity, then the lower (resp. upper) bound of (2.10) cannot be improved.*

Proof. This is an obvious consequence of Proposition 2.2. ■

We now specialize this general setting to the following case.

Definition 2.4. Let $G: \mathcal{R}^d \to \mathcal{R}$ be a continuous absolutely integrable function such that $G(0) = 1$ for which the Fourier transform is non-negative and absolutely integrable. Further, we require that there exist non-negative constants C and κ for which

$$|1 - G(x)| \le C\|x\|^\kappa, \qquad x \in \mathcal{R}^d. \tag{2.17}$$

We let $\mathcal{G}$ denote the class of all such functions G.

Clearly the Gaussian $G(x) = \exp(-\|x\|^2)$ provides an example of such a function. The next lemma mentions some salient properties of $\mathcal{G}$.

Lemma 2.5. *Let $G \in \mathcal{G}$.*

(i) G is a symmetric function, that is

$$G(x) = G(-x), \qquad x \in \mathcal{R}^d. \tag{2.18}$$

(ii)

$$|G(x)| \le 1, \qquad x \in \mathcal{R}^d. \tag{2.19}$$

(iii) G is a positive definite function in the sense of Bochner. In other words, for any real sequence $(y_j)_{j\in\mathcal{Z}^d}$ of finite support, and for any points $(x^j)_{j\in\mathcal{Z}^d}$ in $\mathcal{R}^d$, we have

$$\sum_{j,k\in\mathcal{Z}^d} y_j y_k G(x^j - x^k) \ge 0. \tag{2.20}$$

Proof.

(i) The fact that $\hat{G}$ is real-valued implies the equation

$$2i \int_{\mathcal{R}^d} G(x) \sin x\xi \, dx = \hat{G}(\xi) - \hat{G}(-\xi) \in \mathcal{R}, \qquad \xi \in \mathcal{R}^d,$$

which is a contradiction unless both sides vanish. Thus $\hat{G}$ is a symmetric function. However, G must inherit this symmetry, by the Fourier inversion theorem.

(ii) The non-negativity of $\hat{G}$ provides the relations

$$|G(x)| = \left|(2\pi)^{-d} \int_{\mathcal{R}^d} \hat{G}(\xi) e^{-ix\xi} \, d\xi\right| \le (2\pi)^{-d} \int_{\mathcal{R}^d} \hat{G}(\xi) \, d\xi = G(0) = 1.$$

(iii) The condition $\hat{G} \in L^1(\mathcal{R}^d)$ implies the validity of (2.5) for φ replaced by G, whence

$$\sum_{j,k\in\mathcal{Z}^d} y_j y_k G(x^j - x^k) = (2\pi)^{-d} \int_{\mathcal{R}^d} \Big| \sum_{j\in\mathcal{Z}^d} y_j e^{ix^j\xi} \Big|^2 \hat{G}(\xi)\, d\xi \ge 0,$$

as required. ■

We remark that the first two parts of Lemma 2.5 are usually deduced from the requirement that G be a positive definite function in the Bochner sense (see Katznelson (1976), p.137). We have presented our material in this order because it is the non-negativity condition on $\hat{G}$ which forms our starting point.

We now define the set $\mathcal{A}(G)$ of functions of the form

$$\varphi(x) = c + \int_0^\infty [1 - G(t^{1/2}x)]t^{-1}\, d\alpha(t), \qquad x \in \mathcal{R}^d, \tag{2.21}$$

where c is a constant and $\alpha: [0,\infty) \to \mathcal{R}$ is a non-decreasing function such that

$$\int_1^\infty t^{-1}\, d\alpha(t) < \infty \text{ and } \int_0^1 t^{\kappa/2-1}\, d\alpha(t) < \infty. \tag{2.22}$$

Let us show that (2.21) is well-defined. From (2.19) we have the bound

$$\int_1^\infty \Big|1 - G(t^{1/2}x)\Big| t^{-1}\, d\alpha(t) \le 2 \int_1^\infty t^{-1}\, d\alpha(t) < \infty. \tag{2.22}$$

Moreover, applying condition (2.17) we obtain

$$\int_0^1 \Big|1 - G(t^{1/2}x)\Big| t^{-1}\, d\alpha(t) \le C\|x\|^\kappa \int_0^1 t^{\kappa/2-1}\, d\alpha(t) < \infty. \tag{2.23}$$

Therefore we have shown that the integral of (2.21) is finite and φ is a function of polynomial growth. A simple application of the dominated convergence theorem reveals that φ is also continuous, so that we may view it as a tempered distribution.

The reader will find the following definition convenient.

Definition 2.6. We shall say that a real sequence $(y_j)_{j\in\mathcal{Z}^d}$ of finite support is *zero-summing* if $\sum_{j\in\mathcal{Z}^d} y_j = 0$.

An important property of $\mathcal{A}(G)$ is that it consists of *conditionally negative definite functions of order 1 on* $\mathcal{R}^d$, that is whenever $\varphi \in \mathcal{A}(G)$

$$\sum_{j,k\in\mathcal{Z}^d} y_j y_k \varphi(x^j - x^k) \le 0 \tag{2.24}$$

for every zero-summing sequence $(y_j)_{j\in\mathcal{Z}^d}$ and for any points $(x_j)_{j\in\mathcal{Z}^d}$ in $\mathcal{R}^d$. Indeed, (2.21) provides the equation

$$\sum_{j,k\in\mathcal{Z}^d} y_j y_k \varphi(x^j - x^k) = -\int_0^\infty \sum_{j,k\in\mathcal{Z}^d} y_j y_k G(t^{1/2}(x^j - x^k))\, t^{-1}\, d\alpha(t), \tag{2.25}$$

and the right hand side is non-positive because G is positive definite in the Bochner sense (Lemma 2.5 (iii)).

We now fix attention on a particular element $G \in \mathcal{G}$ and a function $\varphi \in \mathcal{A}(G)$.

Theorem 2.7. *Let $(y_j)_{j\in\mathcal{Z}^d}$ be a zero-summing sequence. Then, for any points $(x^j)_{j\in\mathcal{Z}^d}$ in $\mathcal{R}^d$, we have the equation*

$$\sum_{j,k\in\mathcal{Z}^d} y_j y_k \varphi(x^j - x^k) = -(2\pi)^{-d} \int_{\mathcal{R}^d} \Big| \sum_{j\in\mathcal{Z}^d} y_j e^{ix^j\xi} \Big|^2 H(\xi)\, d\xi, \tag{2.26}$$

where

$$H(\xi) = \int_0^\infty \hat{G}(\xi/t^{1/2}) t^{-d/2-1}\, d\alpha(t), \qquad \xi \in \mathcal{R}^d. \tag{2.27}$$

Furthermore, this latter integral is finite for almost every $\xi \in \mathcal{R}^d$.

Proof. Applying the Fourier inversion theorem to G in (2.25), we obtain

$$\begin{aligned}\sum_{j,k\in\mathcal{Z}^d} y_j y_k \varphi(x^j - x^k) &= -(2\pi)^{-d} \int_0^\infty \int_{[0,2\pi]^d} \Big| \sum_{j\in\mathcal{Z}^d} y_j \exp(it^{1/2}\eta x^j) \Big|^2 \hat{G}(\eta) t^{-1}\, d\eta\, d\alpha(t) \\ &= -(2\pi)^{-d} \int_0^\infty \int_{[0,2\pi]^d} \Big| \sum_{j\in\mathcal{Z}^d} y_j e^{ix^j\xi} \Big|^2 \hat{G}(\xi/t^{1/2}) t^{-d/2-1}\, d\xi\, d\alpha(t),\end{aligned} \tag{2.28}$$

and we have used the substitution $\xi = t^{1/2}\eta$. Because the integrand in the last line is non-negative, we can exchange the order of integration to obtain (2.26). Of course the left hand side of the last equation is finite, so the integrand of (2.25) is an absolutely integrable function, which implies that it is finite almost everywhere. But, by Lemma 2.1, $|\sum_j y_j e^{ix^j\xi}|^2 \neq 0$ for almost every $\xi \in \mathcal{R}^d$ if the sequence $(y_j)_{j\in\mathcal{Z}^d}$ is non-zero. Therefore H is finite almost everywhere. ■

Corollary 2.8. *Given the same hypotheses as Theorem 2.7, we have the equation*

$$F(x) = -(2\pi)^{-d} \int_{\mathcal{R}^d} \Big| \sum_{j\in\mathcal{Z}^d} y_j e^{ix^j\xi} \Big|^2 H(\xi) e^{ix\xi}\, d\xi, \qquad x \in \mathcal{R}^d, \tag{2.29}$$

where F is given by (2.1). Consequently, $\hat{\varphi}(\xi) = -H(\xi)$ for almost every $\xi \in \mathcal{R}^d$, that is

$$\hat{\varphi}(\xi) = -\int_0^\infty \hat{G}(\xi/t^{1/2}) t^{-d/2-1}\, d\alpha(t). \tag{2.30}$$

Proof. It is straightforward to deduce the relation

$$F(x) = -(2\pi)^{-d} \int_0^\infty \int_{[0,2\pi]^d} \Big| \sum_{j\in\mathcal{Z}^d} y_j e^{ix^j\xi} \Big|^2 e^{ix\xi} \hat{G}(\xi/t^{1/2}) t^{-d/2-1}\, d\xi\, d\alpha(t),$$

which is analogous to (2.28). Now the absolute value of this integrand is precisely the integrand in the second line of (2.28). Thus we may apply Fubini's theorem to exchange the order of integration, obtaining (2.29).

Next we prove that $\xi \mapsto -|\sum_j y_j e^{ix^j\xi}|^2 H(\xi)$ is the Fourier transform of F. Indeed, let $\psi: \mathcal{R}^d \to \mathcal{R}$ be an infinitely differentiable function whose partial derivatives enjoy supra-algebraic decay. We must show that

$$\int_{\mathcal{R}^d} \hat{\psi}(x) F(x)\, dx = -\int_{\mathcal{R}^d} \psi(\xi) \Big| \sum_{j\in\mathcal{Z}^d} y_j e^{ix^j\xi} \Big|^2 H(\xi)\, d\xi. \tag{2.31}$$

Applying (2.29) and Fubini's theorem, we get

$$\begin{aligned} \int_{\mathcal{R}^d} \hat{\psi}(x) F(x)\, dx &= -(2\pi)^{-d} \int_{\mathcal{R}^d} \int_{\mathcal{R}^d} \hat{\psi}(x) \Big| \sum_{j\in\mathcal{Z}^d} y_j e^{ix^j\xi} \Big|^2 e^{ix\xi} H(\xi)\, d\xi\, dx \\ &= -\int_{\mathcal{R}^d} \Big| \sum_{j\in\mathcal{Z}^d} y_j e^{ix^j\xi} \Big|^2 H(\xi) \left((2\pi)^{-d} \int_{\mathcal{R}^d} \hat{\psi}(x) e^{ix\xi}\, dx \right) d\xi \\ &= -\int_{\mathcal{R}^d} \Big| \sum_{j\in\mathcal{Z}^d} y_j e^{ix^j\xi} \Big|^2 H(\xi) \psi(\xi)\, d\xi, \end{aligned}$$

which establishes (2.30). However, (2.3) implies that the Fourier transform $\hat{F}(\xi)$ is almost everywhere equal to $|\sum_j y_j e^{ix^j\xi}|^2 \hat{\varphi}(\xi)$. Choosing a nonzero real sequence $(y_j)_{j\in\mathcal{Z}^d}$, we conclude from Lemma 2.1 that $\sum_j y_j e^{ix^j\xi} \neq 0$ for almost all $\xi \in \mathcal{R}^d$, which implies that $\hat{\varphi} = -H$ almost everywhere. ■

3. Pólya frequency functions

For every real sequence $(a_j)_{j=1}^\infty$ such that $\sum_{j=1}^\infty a_j^2 < \infty$, and any non-negative constant γ, we set

$$E(z) = e^{-\gamma z^2} \prod_{j=1}^\infty (1 - a_j^2 z^2), \qquad z \in \mathcal{C}. \tag{3.1}$$

This is an entire function which is nonzero in the vertical strip

$$|\Re z| \le \rho := 1/\sup\{|a_j| : j = 1, 2, \ldots\}.$$

Thus there exists a function $\Lambda: \mathcal{R} \to \mathcal{R}$ such that

$$\int_{\mathcal{R}} \Lambda(t) e^{-zt}\, dt = \frac{1}{E(z)}, \qquad |\Re z| \le \rho. \tag{3.2}$$

This function Λ is what Schoenberg (1951) calls a Pólya frequency function. We have restricted ourselves to Pólya frequency functions Λ which are even, that is

$$\Lambda(t) = \Lambda(-t), \qquad t \in \mathcal{R}. \tag{3.3}$$

Also, it is obvious that

$$\int_{\mathcal{R}} \Lambda(t)\, dt = 1. \tag{3.4}$$

According to (3.1) the Fourier transform of Λ is given by

$$\hat{\Lambda}(\xi) = \frac{1}{E(i\xi)} = \frac{e^{-\gamma\xi^2}}{\prod_{j=1}^{\infty}(1 + a_j^2\xi^2)}, \quad \xi \in \mathcal{R}. \tag{3.5}$$

Thus $\Lambda(\cdot)/\Lambda(0)$ is a member of the set $\mathcal{G}$ described in Definition 2.4 for $d = 1$. Applying Lemma 2.5 we obtain

$$|\Lambda(t)| \le \Lambda(0), \qquad t \in \mathcal{R}. \tag{3.6}$$

However, much more than (3.6) is true. Schoenberg (1951) proved that

$$\det(\Lambda(x_j - y_k))_{j,k=1}^{n} \ge 0 \tag{3.7}$$

whenever $x_1 < \cdots < x_n$ and $y_1 < \cdots < y_n$. This fact will be used in an essential way in Section 4. For the moment we observe that

$$\Lambda(t) \in [0, \Lambda(0)], \qquad t \in \mathcal{R}. \tag{3.8}$$

Let us use $\mathcal{P}$ to denote the class of functions $\Lambda\colon \mathcal{R} \to \mathcal{R}$ that satisfy (3.2) for some $\gamma \ge 0$ and sequence $(a_j)_{j=1}^{\infty}$ satisfying $\sum_{j=1}^{\infty} a_j^2 < \infty$. For any $a > 0$ the function

$$S_a(t) = \frac{1}{2|a|} e^{-|t/a|}, \qquad t \in \mathcal{R}, \tag{3.9}$$

is in $\mathcal{P}$ since

$$\int_{\mathcal{R}} S_a(t) e^{-zt}\, dt = \frac{1}{1 - a^2 z^2}, \qquad |\Re z| < a^{-1}. \tag{3.10}$$

Let $\mathcal{E} = \{S_a : a > 0\}$. These are the only elements of $\mathcal{P}$ that are *not* in $C^2(\mathcal{R})$, because all other members of $\mathcal{P}$ have the property that $\hat{\Lambda}(t) = \mathcal{O}(t^{-4})$ as $|t| \to \infty$. Hence there exists a constant κ such that

$$|\Lambda(0) - \Lambda(t)| \le \kappa t^2, \quad \text{for } t \in \mathcal{R} \text{ and } \Lambda \in \mathcal{P} \setminus \mathcal{E}, \tag{3.11}$$

or

$$|\Lambda(0) - \Lambda(t)| \le \kappa |t|, \quad t \in \mathcal{R}, \quad \Lambda \in \mathcal{E}. \tag{3.12}$$

We note also that every element of $\mathcal{P}$ decays exponentially for large argument (see Karlin (1968), p. 332).

We are now ready to define the multivariate class of functions which interest us. Choose any $\Lambda_1, \ldots, \Lambda_d \in \mathcal{P}$ and define

$$G(x) = \prod_{j=1}^{d} \frac{\Lambda_j(x_j)}{\Lambda_j(0)}, \quad x = (x_1, \ldots, x_d) \in \mathcal{R}^d. \tag{3.13}$$

Clearly, every such G decays exponentially at infinity. Further, according to (3.11) and (3.12), there is a constant $C \geq 0$ such that

$$1 - G(t^{1/2}x) \leq Ct\|x\|_2^2, \tag{3.14}$$

when $\Lambda_j \notin \mathcal{E}$ for every factor Λ_j in (3.11). In the contrary case we only have

$$1 - G(t^{1/2}x) \leq Ct^{1/2}\|x\|_2 + Dt\|x\|_2^2, \tag{3.15}$$

for certain constants C and D. Since the Fourier transform of G is given by

$$\hat{G}(\xi) = \prod_{j=1}^{d} \frac{\hat{\Lambda}_j(\xi_j)}{\Lambda_j(0)}, \qquad \xi = (\xi_1, \ldots, \xi_d) \in \mathcal{R}^d, \tag{3.16}$$

we conclude that G is a member of the class $\mathcal{G}$ of Definition 2.4. We can now construct the set $\mathcal{A}(G)$ for G of the form (3.13). To this end, let $\alpha\colon [0, \infty) \to \mathcal{R}$ be a non-decreasing function such that

$$\int_1^\infty t^{-1}\, d\alpha(t) < \infty, \tag{3.17}$$

and for any constant $c \in \mathcal{R}$ define $\varphi\colon [0, \infty) \to \mathcal{R}$ by (2.21). Thus we see that as long as we require the measure $d\alpha$ to satisfy the extra condition

$$\int_0^1 t^{-1/2}\, d\alpha(t) < \infty \tag{3.18}$$

whenever one of the factors in (3.11) is an element of $\mathcal{E}$, then φ is a continuous function of polynomial growth and the results of Section 2 apply. We let $\mathcal{C}$ denote the class of all functions of the form (2.21) where G is given by (3.13) when α satisfies (3.17) and (3.18).

Let us note that $\mathcal{C}$ contains the following important subclass of functions. In 1938, I. J. Schoenberg proved that a continuous radially symmetric function $\varphi\colon \mathcal{R}^d \to \mathcal{R}$ is conditionally negative definite of order 1 on every $\mathcal{R}^d$ if and only if it has the form

$$\varphi(x) = \varphi(0) + \int_0^\infty \Big(1 - \exp(-t\|x\|^2)\Big) t^{-1}\, d\alpha(t), \qquad x \in \mathcal{R}^d,$$

where $\alpha\colon [0,\infty) \to \mathcal{R}$ is a non-decreasing function satisfying (3.17). Now $G(x) =$ is clearly of the form (3.13), implying that we do indeed have a subclass of $\mathcal{C}$. It is this subclass of $\mathcal{C}$ which is studied in Baxter (1991). Further, we have established Theorem 2.7 and Corollary 2.8 under weaker conditions than those assumed in Baxter (1991).

Our class $\mathcal{C}$ also contains functions of the form

$$\varphi(x) = c + \int_0^\infty \Big(1 - \exp(-t^{1/2}\|x\|_1)\Big) t^{-1}\, d\alpha(t), \qquad x \in \mathcal{R}^d,$$

where $\alpha\colon [0,\infty) \to \mathcal{R}$ is a non-decreasing function satisfying (3.17) and (3.18), and $\|x\|_1 = \sum_{j=1}^d |x_j|$ for $x = (x_1, \ldots, x_d) \in \mathcal{R}^d$. For instance, using the easily verified formula

$$\frac{\gamma}{\Gamma(1+2\gamma)} \int_0^\infty \Big(1 - e^{-t^{1/2}\sigma}\Big) t^{\gamma-1} e^{-\delta t}\, dt = \delta^{-2\gamma} - (\delta+\sigma)^{-\gamma}, \qquad \gamma > 0,$$

which is valid for $\delta > 0$ and $\gamma > -1/2$ or $\delta = 0$ and $-1/2 < \gamma < 0$, we see that either $\varphi(x) = \|x\|_1^\tau$, for $0 < \tau < 1$, or $\varphi(x) = \tau(\delta + \|x\|_1)^\tau$, for $\delta > 0$ and $\tau < 1$, are in our class $\mathcal{C}$.

Let us now discuss some additional properties of the Fourier transform of a function $\varphi \in \mathcal{C}$. First, observe that (3.5) implies that $\hat{\Lambda}$ is a decreasing function on $[0,\infty)$ for every Λ in $\mathcal{P}$. Consequently every $G \in \mathcal{G}$ satisfies the inequality $\hat{G}(\xi) \le \hat{G}(\eta)$ for $\xi \ge \eta \ge 0$. This property is inherited by the function H of (2.27), that is

$$H(\xi) \le H(\eta) \text{ whenever } \xi \ge \eta \ge 0, \tag{3.19}$$

which allows us to strengthen Theorem 2.7.

Proposition 3.1. *For every $G \in \mathcal{G}$, the function H given by (2.27) is continuous on $(\mathcal{R} \setminus \{0\})^d$.*

Proof. We first show that H is finite on $(\mathcal{R} \setminus \{0\})^d$. From Corollary 2.8, we know that $\hat{\varphi} = -H$ almost everywhere, which implies that every set of positive measure contains a point at which H is finite. In particular, let δ be a positive number and set $U_\delta = \{\xi \in \mathcal{R}^d : 0 < \xi_j < \delta, \quad j = 1, \ldots, d\}$. Thus there is a point $\eta \in U_\delta$ such that $H(\eta) < \infty$. Applying (3.19) and recalling that H is a symmetric function, we deduce the inequality

$$H(\xi) \le H(\eta) < \infty, \qquad \xi \in F_\delta, \tag{3.20}$$

where $F_\delta := \{\xi \in \mathcal{R}^d : |\xi_j| \ge \delta, \quad j = 1, \ldots, d\}$. Since $\delta > 0$ is arbitrary, we see that H is finite in $(\mathcal{R} \setminus \{0\})^d$.

To prove that H is continuous in F_δ, let $(\xi_n)_{n=1}^\infty$ be a convergent sequence in F_δ with limit ξ_∞. By (3.20), the functions

$$\{t \mapsto \hat{G}(\xi_n t^{-1/2}) t^{-d/2-1} : n = 1, 2, \ldots\}$$

are absolutely integrable on $[0,\infty)$ with respect to the measure $d\alpha$. Moreover, they are dominated by the $d\alpha$-integrable function $t \mapsto \hat{G}(\eta t^{-1/2})t^{-d/2-1}$. Finally, the continuity of $\hat{G}$ provides the equation

$$\lim_{n\to\infty} \hat{G}(\xi_n t^{-1/2})t^{-d/2-1} = \hat{G}(\xi_\infty t^{-1/2})t^{-d/2-1}, \qquad t \in [0,\infty),$$

and thus $\lim_{n\to\infty} H(\xi_n) = H(\xi_\infty)$ by the dominated convergence theorem. Since δ was an arbitrary positive number, we conclude that H is continuous on $(\mathcal{R} \setminus \{0\})^d$. ■

The remainder of this section requires a distinction of cases. The first case (Case I) is the nicest. This occurs when every factor Λ_j in (3.13) has a positive exponent γ_j in the Fourier transform formula (3.5). We let Case II denote the contrary case.

For Case I we have the bound

$$\hat{G}(\xi) \le A\exp(-B\|x\|^2), \qquad \xi \in \mathcal{R}^d,$$

for some positive constants A and B, which implies the limit

$$\lim_{t\to 0} \hat{G}(\xi t^{-1/2})t^{-d/2-1} = 0, \qquad \xi \ne 0.$$

Thus the function $t \mapsto \hat{G}(\xi t^{-1/2})t^{-d/2-1}$ is continuous for $t \in [0,\infty)$ when ξ is nonzero, which implies that

$$\int_0^1 \hat{G}(\xi t^{-1/2})t^{-d/2-1}\, d\alpha(t) < \infty, \qquad \xi \ne 0.$$

Moreover, since

$$\int_1^\infty \hat{G}(\xi t^{-1/2})t^{-d/2-1}\, d\alpha(t) \le A\int_1^\infty t^{-1}\, d\alpha(t) < \infty,$$

we have $H(\xi) < \infty$ for every $\xi \in \mathcal{R}^d \setminus \{0\}$. A simple extension of the proof of Proposition 3.1 shows that H is continuous on $\mathcal{R}^d \setminus \{0\}$. Furthermore, we can prove that for $H \in C^\infty(\mathcal{R}^d \setminus \{0\})$ in Case I. For this purpose, we note that it is sufficient to show that every derivative of $\hat{G}(\xi t^{-1/2})t^{-d/2-1}$ with respect to ξ is an absolutely integrable function with respect to the measure $d\alpha$ on $[0,\infty)$, because then we are justified in differentiating under the integral sign in (2.27). By Definition 3.13, we only need to show that every derivative of $\hat{\Lambda}$, where $\hat{\Lambda}$ is given by (3.5) with $\gamma > 0$, enjoys faster than algebraic decay for large argument. To this end we claim that for every $C < 1/\sup\{|a_j| : j = 1,2,\ldots\}$ there is a constant D such that

$$\left|\hat{\Lambda}(\xi + i\eta)\right| \le De^{-\gamma\xi^2}, \qquad \xi \in \mathcal{R}, \quad |\eta| \le C. \tag{3.21}$$

To verify the claim, observe that when $|\eta| \le C \le |\xi|$ we have the inequalities

$$\left|e^{-\gamma(\xi+i\eta)^2}\right| \le e^{C^2\gamma} e^{-\gamma\xi^2} \text{ and } |1 + a_j^2(\xi + i\eta)^2| \ge 1 + a_j^2(\xi^2 - \eta^2) \ge 1.$$

Thus, setting $M = \max\{|\hat{\Lambda}(\xi + i\eta)|e^{\gamma\xi^2} : |\xi| \le C, |\eta| \le C\}$, we conclude that $D := \max\{M, e^{C^2\gamma}\}$ is suitable in (3.21). Now, we apply the Cauchy integral formula to estimate the kth derivative

$$\hat{\Lambda}^{(k)}(\xi) = \frac{1}{2\pi i k!} \int_\Gamma \frac{\hat{\Lambda}(\zeta)}{(\zeta - \xi)^{k+1}} \, d\zeta,$$

where $\Gamma : [0, 2\pi] \to \mathcal{C}$ is given by $\Gamma(t) = re^{it}$ and $r < C$ is a constant to obtain the bound

$$\left|\hat{\Lambda}^{(k)}(\xi)\right| \le (D/\alpha^k) e^{-\gamma \min\{(\xi-r)^2, (\xi+r)^2\}}, \qquad \xi \in \mathcal{R},$$

and the desired supra-algebraic decay. We now state this formally.

Proposition 3.2. *In Case I, the function H of (2.27) is a member of $C^\infty(\mathcal{R}^d \setminus \{0\}$. Moreover $H = -\hat{\varphi}$ on $\mathcal{R}^d \setminus \{0\}$.*

Proof. It remains to identify $-H$ with $\hat{\varphi}$ on $\mathcal{R}^d \setminus \{0\}$. Let $\psi\colon \mathcal{R}^d \to \mathcal{R}$ be an infinitely differentiable function whose support is a compact subset of $\mathcal{R}^d \setminus \{0\}$. By definition of $\hat{\varphi}$ we have

$$\langle \hat{\varphi}, \psi \rangle = \int_{\mathcal{R}^d} \hat{\psi}(x)\varphi(x)\, dx, \tag{3.22}$$

where $\langle \hat{\varphi}, \psi \rangle$ denotes the value of a tempered distribution $\hat{\varphi}$ on a test function ψ. Substituting the expression for φ given by (2.21) into the right hand side of (3.22) and using the fact that

$$0 = \psi(0) = (2\pi)^{-d} \int_{\mathcal{R}^d} \hat{\psi}(\xi)\, d\xi \tag{3.23}$$

gives

$$\langle \hat{\varphi}, \psi \rangle = -\int_{\mathcal{R}^d} \left(\int_0^\infty \hat{\psi}(x)(1 - G(t^{1/2}x))t^{-1}\, d\alpha(t) \right) dx.$$

We want to swap the order of integration here. This will be justified by Fubini's theorem if we can show that

$$\int_{\mathcal{R}^d} \left(\int_0^\infty |\hat{\psi}(x)|(1 - G(t^{1/2}x))t^{-1}\, d\alpha(t) \right) dx < \infty. \tag{3.24}$$

We defer the proof of (3.24) and press on. Swapping the order of integration and recalling (3.23) yields

$$\begin{aligned}
\langle \hat{\varphi}, \psi \rangle &= -\int_0^\infty \left(\int_{\mathcal{R}^d} \hat{\psi}(x) G(t^{1/2}x)\, dx \right) t^{-1}\, d\alpha(t) \\
&= -\int_0^\infty \left(\int_{\mathcal{R}^d} \psi(\xi)\hat{G}(\xi t^{-1/2})\, d\xi \right) t^{-d/2-1}\, d\alpha(t)
\end{aligned}$$

using Parseval's relation in the last line. Once again, we want to swap the order of integration and, as before, this is justified by Fubini's theorem if a certain integral is finite, specifically

$$\int_0^\infty \left(\int_{\mathcal{R}^d} |\psi(\xi)|\, \hat{G}(\xi t^{-1/2})\, d\xi \right) t^{-d/2-1}\, d\alpha(t) < \infty. \tag{3.25}$$

The proof of (3.25) will also be found in Lemma 3.3 below. After swapping the order of integration we have

$$\langle \hat{\varphi}, \psi \rangle = -\int_{\mathcal{R}^d} \psi(\xi) H(\xi)\, d\xi, \tag{3.26}$$

which implies that $\hat{\varphi} = -H$ in $\mathcal{R}^d \setminus \{0\}$.

Our final task is to show that inequalities (3.24) and (3.25) are valid. For (3.24), we have by (3.14) and the fact that G is nonnegative

$$\begin{aligned}
&\int_{\mathcal{R}^d} \left(\int_0^\infty |\hat{\psi}(x)|(1 - G(t^{1/2}x)) t^{-1}\, d\alpha(t) \right) dx \\
&\le \int_{\mathcal{R}^d} \left(\kappa \int_0^1 |\hat{\psi}(x)| \|x\|^2\, d\alpha(t) \right) dx + \int_{\mathcal{R}^d} \left(\int_1^\infty |\hat{\psi}(x)| t^{-1}\, d\alpha(t) \right) dx \\
&= \kappa(\alpha(1) - \alpha(0)) \int_{\mathcal{R}^d} |\hat{\psi}(x)| \|x\|^2\, dx + \left(\int_1^\infty t^{-1}], d\alpha(t) \right) \left(\int_{\mathcal{R}^d} |\hat{\psi}(x)|\, dx \right) \\
&< \infty,
\end{aligned}$$

since $\hat{\psi}$ must enjoy faster than algebraic decay because ψ is an infinitely differentiable function.

For (3.25), the substitution $\eta = \xi t^{-1/2}$ provides the integral

$$I := \int_0^\infty \left(\int_{\mathcal{R}^d} |\psi(\eta t^{1/2})| \hat{G}(\eta)\, d\eta \right) t^{-1}\, d\alpha(t).$$

Now there is a constant D such that $|\psi(y)| \le D\|y\|^2$ for every $y \in \mathcal{R}^d$, because the support of ψ is a closed subset of $\mathcal{R}^d \setminus \{0\}$. Hence

$$\begin{aligned}
I &\le \int_0^1 D \left(\int_{\mathcal{R}^d} \hat{G}(\eta) \|\eta\|^2\, d\eta \right) d\alpha(t) + (2\pi)^d G(0) \|\psi\|_\infty \int_1^\infty t^{-1}\, d\alpha(t) \\
&< \infty.
\end{aligned}$$

The proof is complete. ■

4. Lower bounds on eigenvalues

Let $\varphi\colon \mathcal{R}^d \to \mathcal{R}$ be a member of $\mathcal{C}$ and let $(y_j)_{j\in\mathcal{Z}^d}$ be a zero-summing sequence. An immediate consequence of (2.26) is the equation

$$\sum_{j,k\in\mathcal{Z}^d} y_j y_k \varphi(j-k) = (2\pi)^{-d} \int_{\mathcal{R}^d} \Big| \sum_{j\in\mathcal{Z}^d} y_j e^{ij\xi} \Big|^2 \hat{\varphi}(\xi)\, d\xi, \tag{4.1}$$

where $\hat{\varphi}(\xi) = -H(\xi)$ for almost all $\xi \in \mathcal{R}^d$ and H is given by (2.27). Moreover, (2.6) is valid, that is

$$\sum_{j,k \in \mathcal{Z}^d} y_j y_k \varphi(j-k) = (2\pi)^{-d} \int_{[0,2\pi]^d} \Big| \sum_{j \in \mathcal{Z}^d} y_j e^{ij\xi} \Big|^2 \sigma(\xi)\, d\xi, \tag{4.2}$$

where σ is given by (2.7). Applying (2.30), we have

$$\begin{aligned} |\sigma(\xi)| &= \sum_{k \in \mathcal{Z}^d} \Big| \hat{\varphi}(\xi + 2\pi k) \Big| \\ &= \int_0^\infty \sum_{k \in \mathcal{Z}^d} \hat{G}(t^{-1/2}(\xi + 2\pi k))\ t^{-d/2-1}\, d\alpha(t). \end{aligned} \tag{4.3}$$

As in Section 2, we consider essential upper and lower bounds on σ. Let us begin this study by fixing $t > 0$ and analysing the function

$$\tau(\xi) = \sum_{k \in \mathcal{Z}^d} \hat{G}(t^{-1/2}(\xi + 2\pi k)), \qquad \xi \in \mathcal{R}^d. \tag{4.4}$$

By (3.14), we have

$$\tau(\xi) = \prod_{j=1}^d \frac{E_j(\xi_j)}{\Lambda_j(0)}, \qquad \xi \in \mathcal{R}^d, \tag{4.5}$$

where

$$E_j(x) = \sum_{k \in \mathcal{Z}} \hat{\Lambda}_j((x + 2\pi k)t^{-1/2}), \qquad x \in \mathcal{R}, \quad j = 1, \ldots, d. \tag{4.6}$$

We now employ the following key lemma.

Lemma 4.1. *Let $\Lambda \in \mathcal{P}$ and let*

$$E(x) = \sum_{k \in \mathcal{Z}} \hat{\Lambda}((x + 2\pi k)t^{-1/2}), \qquad x \in \mathcal{R}.$$

Then E is an even function and $E(0) \geq E(x) \geq E(y) \geq E(\pi)$ for every x and y in $\mathcal{R}$ with $0 \leq x \leq y \leq \pi$.

Proof. The exponential decay of Λ and the absolute integrability of $\hat{\Lambda}$ imply that the Poisson summation formula is valid, which gives the relation

$$E(x) = t^{1/2} \sum_{k \in \mathcal{Z}} \Lambda(kt^{1/2}) e^{ikx}, \qquad x \in \mathcal{R}. \tag{4.7}$$

Now the sequence $a_k := \Lambda(kt^{1/2})$, $k \in \mathcal{Z}$, is an even, exponentially decaying Pólya frequency sequence, that is every minor of the Toeplitz matrix $(a_{j-k})_{j,k \in \mathcal{Z}}$ is non-negative definite (and we see that this is a consequence of (3.7)). By a result of Edrei (1953), $\sum_{k \in \mathcal{Z}} a_k z^k$ is a meromorphic function on an annulus $\{z \in \mathcal{C} : 1/R \le |z| \le R\}$, for some $R > 1$, and enjoys an infinite product expansion of the form

$$\sum_{k\in\mathcal{Z}} a_k z^k = Ce^{\lambda(z+z^{-1})} \prod_{j=1}^{\infty} \frac{(1+\alpha_j z)(1+\alpha_j z^{-1})}{(1-\beta_j z)(1-\beta_j z^{-1})}, \qquad z \ne 0, \tag{4.8}$$

where $C \ge 0$, $\lambda \ge 0$, $0 < \alpha_j, \beta_j < 1$ and $\sum_{j=1}^{\infty} \alpha_j + \beta_j < \infty$. Hence

$$E(x) = Ct^{1/2} e^{2\lambda \cos x} \prod_{j=1}^{\infty} \frac{1 + 2\alpha_j \cos x + \alpha_j^2}{1 - 2\beta_j \cos x + \beta_j^2}, \qquad x \in \mathcal{R}. \tag{4.9}$$

Observe that each term in the product is an even function which is decreasing on $[0, 2\pi]$, which provides the required inequality. ■

In particular, $E_j(x) \ge E_j(\pi)$ for $j = 1, \ldots, d$, where E_j is given by (4.6). Hence

$$\tau(\xi) \ge \tau(\pi e), \qquad \xi \in \mathcal{R}^d, \tag{4.10}$$

and applying (4.3) we get

$$|\sigma(\xi)| \ge |\sigma(\pi)|, \qquad \xi \in \mathcal{R}^d. \tag{4.11}$$

We now come to our principal result.

Theorem 4.2. *Let $(y_j)_{j\in\mathcal{Z}^d}$ be a zero-summing sequence and let $\varphi \in \mathcal{C}$. Then we have the inequality*

$$\Big| \sum_{j,k\in\mathcal{Z}^d} y_k y_k \varphi(j-k) \Big| \ge |\sigma(\pi e)| \sum_{j\in\mathcal{Z}^d} y_j^2. \tag{4.12}$$

Proof. Equation (4.2) and the Parseval relation provide the inequality

$$\Big| \sum_{j,k\in\mathcal{Z}^d} y_k y_k \varphi(j-k) \Big| \ge |\sigma(\pi e)| (2\pi)^{-d} \int_{[0,2\pi]^d} \Big| \sum_{j\in\mathcal{Z}^d} y_j e^{ij\xi} \Big|^2 d\xi = |\sigma(\pi e)| \sum_{j\in\mathcal{Z}^d} y_j^2,$$

as in inequality (2.10). ■

Of course, we are interested in showing that (4.12) cannot be improved, that is $|\sigma(\pi e)|$ cannot be replaced by a larger number independent of $(y_j)_{j\in\mathcal{Z}^d}$. Recalling Proposition 2.2, this is true if σ is continuous at πe. In fact, we can use Lemma 4.1 to prove that σ is continuous everywhere in the set $(0, 2\pi)^d$. We first collect some necessary preliminary results.

Lemma 4.3. *The function τ given by (4.4) is continous for every $t > 0$ and satisfies the inequality*

$$\tau(\xi) \le \tau(\eta) \text{ for } 0 \le \eta \le \xi \le \pi e. \tag{4.13}$$

Furthermore,

$$\tau(\pi e + \xi) = \tau(\pi e - \xi) \text{ for all } \xi \in (-\pi, \pi)^d. \tag{4.14}$$

Proof. The definition of G, (4.5) and (4.7) provide the Fourier series

$$\tau(\xi) = t^{d/2} \sum_{k \in \mathcal{Z}^d} G(kt^{1/2}) e^{ik\xi}, \qquad \xi \in \mathcal{R}^d, \tag{4.15}$$

and the exponential decay of G implies the uniform convergence of this series. Hence τ is continuous, being the uniform limit of the finite sections of (4.15).

Applying the product formula (4.5) and Lemma 4.1, we obtain (4.13) and (4.14). ∎

Proposition 4.4. *σ is continuous on $(0, 2\pi)^d$.*

Proof. Equation (4.2) implies that $\left|\sum_{j \in \mathcal{Z}^d} y_j e^{ij\xi}\right|^2 |\sigma(\xi)| < \infty$ for almost every $\xi \in [0, 2\pi]^d$. Consequently, σ is finite almost everywhere, by Lemma 2.1. Thus every non-empty open subset of $[0, 2\pi]^d$ contains a point at which σ is finite. Specifically, let $\delta \in (0, \pi)$ and define the closed set $K_\delta := [\delta, 2\pi - \delta]^d$. Thus the open set $[0, 2\pi]^d \setminus K_\delta$ contains a point, η say, for which

$$\infty > |\sigma(\eta)| = \int_0^\infty \sum_{k \in \mathcal{Z}^d} \hat{G}((\eta + 2\pi k)t^{-1/2})\, t^{-d/2-1}\, d\alpha(t). \tag{4.16}$$

Let us show that σ is continuous in K_δ. To this end, choose any convergent sequence $(\xi_n)_{n=1}^\infty$ in K_δ and let ξ_∞ denote its limit. We must prove that $\lim_{n\to\infty} \sigma(\xi_n) = \sigma(\xi_\infty)$. Now Lemma 4.3 and (4.16) supply the bound

$$|\sigma(\xi_n)| \le |\sigma(\eta)| < \infty, \qquad n = 1, 2, \ldots,$$

that is the functions

$$\{t \mapsto \sum_{k \in \mathcal{Z}^d} \hat{G}((\xi_n + 2\pi k)t^{-1/2})t^{-d/2-1}\, d\alpha(t) : n = 1, 2, \ldots\}$$

are absolutely integrable on $[0, \infty)$ with respect to the measure $d\alpha$. Moreover, they are dominated by the absolutely integrable function $t \mapsto \sum_{k \in \mathcal{Z}^d} \hat{G}((\eta + 2\pi k)t^{-1/2})t^{-d/2-1}$. However, the continuity of τ proved in Lemma 4.3 allows to deduce that

$$\lim_{n\to\infty} \sum_{k \in \mathcal{Z}^d} \hat{G}((\xi_n + 2\pi k)t^{-1/2})t^{-d/2-1} = \sum_{k \in \mathcal{Z}^d} \hat{G}((\xi_\infty + 2\pi k)t^{-1/2})t^{-d/2-1}, \qquad t > 0.$$

Thus the dominated convergence theorem implies that $\lim_{n\to\infty} \sigma(\xi_n) = \sigma(\xi_\infty)$. Since $\delta \in (0, \pi)$ was arbitrary, we conclude that σ is continuous in all of $(0, 2\pi)^d$. ∎

Corollary 4.5. *Inequality (4.12) cannot be improved for $\varphi \in \mathcal{C}$.*

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