1.

(a) If $S_t = e^{\sigma W_t}$ then

$$\mathbb{E}\, S_t S_T = \mathbb{E}\, e^{\sigma(W_t + W_T)}$$

$$= \mathbb{E}\, e^{\sigma(2W_t + W_T - W_t)}.$$

Now W_t and $W_T - W_t$ are independent, by the BM axioms, and $W_t \sim N(0,t)$, $W_T - W_t \sim N(0, T-t)$, so

$$\mathbb{E}\, S_t S_T = \mathbb{E}\, e^{2\sigma W_t} \; \mathbb{E}\, e^{\sigma(W_T - W_t)}$$

$$= e^{2\sigma^2 t} \cdot e^{\frac{1}{2}\sigma^2(T-t)}$$

$$= e^{\frac{1}{2}\sigma^2 T} \; e^{\frac{3}{2}\sigma^2 t}$$

$$= e^{\frac{1}{2}\sigma^2(T+3t)},$$

using $\mathbb{E}\, e^{cZ} = e^{c^2/2}$ for $Z \sim N(0,1)$, $c \in \mathbb{R}$.

(b) $$\mathbb{E}\, A_T = \frac{1}{T}\int_0^T \mathbb{E}\, S_t \, dt$$

$$= \frac{1}{T}\int_0^T \mathbb{E}\, e^{\sigma W_t} \, dt$$

$$= \frac{1}{T}\int_0^T e^{\sigma^2 t/2} \, dt$$

$$= \frac{1}{T}\left[\frac{e^{\sigma^2 t/2}}{\sigma^2/2} \right]_{t=0}^{T}$$

$$= \frac{e^{\sigma^2 T/2} - 1}{\sigma^2 T/2}.$$

Now

$$\mathbb{E} S_T A_T = \frac{1}{T} \int_0^T \mathbb{E}\, S_T S_t \; dt$$

$$= \frac{1}{T} \int_0^T e^{\sigma^2 (T + 3t)/2} \; dt \qquad \text{(by 1(a))}$$

$$= \frac{e^{\sigma^2 T/2}}{T} \left[\frac{e^{3\sigma^2 t/2}}{3\sigma^2/2} \right]_{t=0}^{T}$$

$$= \frac{e^{\sigma^2 T/2}}{(3\sigma^2 T/2)} \left[e^{3\sigma^2 T/2} - 1 \right]$$

$$= \frac{e^{\sigma^2 T} - e^{\sigma^2 T/2}}{3\sigma^2 T/2}.$$

(c) $W_{3,t} = \cos\theta \; W_{1,t} + \sin\theta \; W_{2,t}$

$\equiv c\, W_{1,t} + s\, W_{2,t}.$

Then

$$\mathbb{E}\, Z_t Z_t^T = \begin{pmatrix} \mathbb{E}\, W_{1,t}^2 & \mathbb{E}\, W_{1,t} W_{3,t} \\ \mathbb{E}\, W_{1,t} W_{3,t} & \mathbb{E}\, W_{3,t}^2 \end{pmatrix},$$

Now $\mathbb{E}\, W_{1,t}^2 = t$ and

$$\mathbb{E}\, W_{3,t}^2 = \mathbb{E}\left(c^2 W_{1,t}^2 + 2cs\, W_{1,t} W_{2,t} + s^2 W_{2,t}^2\right)$$

$$= (c^2 + s^2)t$$

$$= t$$

since

$$\mathbb{E}\, W_{1,t} W_{2,t} = \mathbb{E} W_{1,t} \cdot \mathbb{E}\, W_{2,t} = 0,$$

because $W_{1,t}$ and $W_{2,t}$ are independent BMs.

Finally,

$$\mathbb{E}\, W_{1,t} W_{3,t} = \mathbb{E}\, W_{1,t}(cW_{1,t} + sW_{2,t})$$

$$= ct.$$

Thus

$$\mathbb{E}\, Z_t Z_t^T = t\begin{pmatrix} 1 & c \\ c & 1 \end{pmatrix}$$

$$= t \underbrace{\begin{pmatrix} 1 & \cos\theta \\ \cos\theta & 1 \end{pmatrix}}_{M}.$$

(d) We need

$$\begin{pmatrix} a & b \\ b & a \end{pmatrix}^2 = \begin{pmatrix} 1 & c \\ c & 1 \end{pmatrix}$$

or $$\begin{pmatrix} a^2+b^2 & 2ab \\ 2ab & a^2+b^2 \end{pmatrix} = \begin{pmatrix} 1 & c \\ c & 1 \end{pmatrix},$$

Thus $a^2+b^2=1$, which implies that we can write
$$a = \cos\phi \quad \text{and} \quad b = \sin\phi.$$

Hence
$$\cos\theta = 2ab = 2\cos\phi\sin\phi = \sin 2\phi$$

or
$$\cos\theta = \sin\left(\tfrac{\pi}{2}-\theta\right) = \sin 2\phi$$

i.e.
$$\tfrac{\pi}{2}-\theta = 2\phi$$

or
$$\phi = \tfrac{1}{2}\left(\tfrac{\pi}{2}-\theta\right).$$

In the question I state that $0<\theta<\frac{\pi}{2}$, so $0<\phi<\frac{\pi}{4}$ and $\cos\phi>0$.

Further,
$$\det\begin{pmatrix} a & b \\ b & a \end{pmatrix} = a^2-b^2 = \cos^2\phi-\sin^2\phi$$
$$= \cos 2\phi$$
$$= \sin\theta$$

which is positive if $0<\theta<\pi/2$. Thus

$M^{\frac{1}{2}} = \begin{pmatrix} \cos\phi & \sin\phi \\ \sin\phi & \cos\phi \end{pmatrix}$ <u>is</u> pos def when $\phi = \frac{1}{2}\left(\frac{\pi}{2}-\theta\right)$ and $0<\theta<\frac{\pi}{2}$.

2.

(a) $X_t = \cos(W_t - c)$, $c \in \mathbb{R}$ constant.

Thus $X_t = f(W_t)$ where $f(x) = \cos(x - c)$

and, by Itô's Lemma,

$$dX_t = f'(W_t)\, dW_t + \frac{1}{2} f''(W_t)\, dt$$

or

$$dX_t = -\sin(W_t - c)\, dW_t - \frac{1}{2}\cos(W_t - c)\, dt.$$

Then $m(t) = \mathbb{E}X_t$ satisfies, on taking expectations of the SDE,

$$\frac{dm}{dt} = -\frac{1}{2} m$$

or

$$m(t) = e^{-\frac{1}{2}t}\, m(0).$$

Now $m(0) = \mathbb{E}\cos(-c) = \cos c$, since $W_0 = 0$,

so

$$\mathbb{E}\cos(W_t - c) = \cos c \,.\, e^{-t/2}.$$

(b) If $X_t = f(W_t)$ then Itô's Lemma states

that

$$dX_t = f'(W_t)\, dW_t + \frac{1}{2} f''(W_t)\, dt.$$

If $M(t) = \mathbb{E}X_t = \mathbb{E}f(W_t)$ then

$$\frac{dm}{dt} = \frac{1}{2}\,\mathbb{E} f''(W_t)$$

or $$\frac{d}{dt}\,\mathbb{E} f(W_t) = \frac{1}{2}\,\mathbb{E} f''(W_t).$$

Finally, if $f(x) = x^4$, then $f''(x) = 12x^2$ and

$$\frac{d}{dt}\,\mathbb{E}(W_t^4) = \frac{1}{2}\cdot 12\,\mathbb{E}W_t^2 = 6t$$

so that

$$\mathbb{E}(W_t^4) = 3t^2 + \mathbb{E}(W_0^4) = 3t^2.$$

However, if $f(x) = x^6$, then $f''(x) = 30x^4$ and

$$\frac{d}{dt}\,\mathbb{E}(W_t^6) = \frac{1}{2}\cdot 30\,\mathbb{E}\,W_t^4$$

$$= 15\,\mathbb{E}\,W_t^4$$

$$= 45\,t^2,$$

so $$\mathbb{E}(W_t^6) = 15t^3 + \mathbb{E}\,W_0^6 = 15t^3.$$

(c) $X \sim N(\mu, \sigma^2)$ so $X = \mu + \sigma Z$, $Z \sim N(0,1)$.

Then

$$\mathbb{E}\left(e^{\sigma X} - 1\right)_+ = \mathbb{E}\left(e^{\mu + \sigma Z} - 1\right)_+$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \left(e^{\mu + \sigma s} - 1\right)_+ e^{-s^2/2} \, ds .$$

Now $e^{\mu+\sigma s} - 1 \geq 0$ if and only if $\mu + \sigma s \geq 0$, i.e. $s \geq -\frac{\mu}{\sigma}$. Thus

$$\mathbb{E}\left(e^{\sigma X} - 1\right)_+ = \frac{1}{\sqrt{2\pi}} \int_{-\mu/\sigma}^{\infty} \left(e^{\mu+\sigma s} - 1\right) e^{-s^2/2} \, ds$$

$$= I_1 - I_2$$

where

$$I_1 = \frac{1}{\sqrt{2\pi}} \int_{-\mu/\sigma}^{\infty} e^{\mu+\sigma s} e^{-s^2/2} \, ds$$

and

$$I_2 = \frac{1}{\sqrt{2\pi}} \int_{-\mu/\sigma}^{\infty} e^{-s^2/2} \, ds.$$

Thus $I_2 = 1 - \Phi\left(-\frac{\mu}{\sigma}\right) = \Phi\left(\frac{\mu}{\sigma}\right)$.

For I_1, we have, completing the square,

$$I_1 = \frac{e^{\mu}}{\sqrt{2\pi}} \int_{-\mu/\sigma}^{\infty} e^{-\frac{1}{2}(s^2 - 2\sigma s + \sigma^2 - \sigma^2)} \, ds$$

$$= e^{\mu + \frac{\sigma^2}{2}} \cdot \frac{1}{\sqrt{2\pi}} \int_{-\frac{\mu}{\sigma}}^{\infty} e^{-\frac{1}{2}(s-\sigma)^2} \, ds$$

$$= e^{\mu + \frac{\sigma^2}{2}} \cdot \frac{1}{\sqrt{2\pi}} \int_{-\frac{\mu}{\sigma} - \sigma}^{\infty} e^{-\frac{1}{2}u^2} \, du$$

$(u = s - \sigma)$

$$= e^{\mu + \frac{\sigma^2}{2}} \left(1 - \Phi\left(-\frac{\mu}{\sigma} - \sigma\right)\right)$$

$$= e^{\mu + \frac{\sigma^2}{2}} \Phi\left(\frac{\mu}{\sigma} + \sigma\right).$$

Hence

$$\mathbb{E}(e^X - 1)_+ = e^{\mu + \frac{\sigma^2}{2}} \Phi\left(\frac{\mu}{\sigma} + \sigma\right) - \Phi\left(\frac{\mu}{\sigma}\right).$$

3.

(a) If $f(S,t) = S^2 E$, where $E = e^{3r(T-t)}$,

then
$$\frac{\partial f}{\partial S} = 2SE, \quad \frac{\partial^2 f}{\partial S^2} = 2E$$

and
$$\frac{\partial f}{\partial t} = -3r\, S^2 E.$$

Hence
$$-rf + rS\frac{\partial f}{\partial S} + rS^2\frac{\partial^2 f}{\partial S^2} + \frac{\partial f}{\partial t}$$

$$= -rS^2E + 2rS^2E + 2rS^2E - 3rS^2E$$

$$= S^2E(-r + 2r + 2r - 3r)$$

$$= 0, \quad \text{as required.}$$

(b) If $f(S,t) = V(t)$, then $\dfrac{\partial f}{\partial S} = \dfrac{\partial^2 f}{\partial S^2} = 0$

and
$$0 = -rf + \frac{\partial f}{\partial t} = -rV + \frac{dV}{dt}.$$

Hence
$$V(t) = Ce^{rt}.$$

If $V(T) = K$, then $Ce^{rT} = K$

and $V(t) = Ke^{-rT}e^{rt} = Ke^{-r(T-t)}$.

(c) If $f_1(S,t) = S^2 e^{3r(T-t)}$,

then f_1 satisfies Black–Scholes and

$$f_1(S,T) = S^2.$$

Further, if $f_2(S,t) = K e^{-r(T-t)}$,

then $f_2(S,T) = K$ and f_2 also satisfies Black–Scholes.

Since Black–Scholes is a linear PDE,

$$f(S,t) = f_1(S,t) - f_2(S,t)$$

satisfies Black–Scholes and

$$f(S_T,T) = f_1(S_T,T) - f_2(S_T,T)$$

$$= S_T^2 - K.$$

(d) This is bookwork:

$$\frac{\partial f}{\partial x} = \frac{\partial f}{\partial S} \cdot \frac{dS}{dx} = \frac{\partial f}{\partial S} \cdot S$$

or

$$\frac{\partial}{\partial x} = S \frac{\partial}{\partial S} . \qquad (1)$$

Hence

$$\frac{\partial^2}{\partial x^2} = S \frac{\partial}{\partial S}\left(S \frac{\partial}{\partial S} \right)$$

$$= S\left(\frac{\partial}{\partial S} + S \frac{\partial^2}{\partial S^2} \right)$$

$$= \underbrace{S \frac{\partial}{\partial S}}_{\partial/\partial x \text{ by } (1)} + S^2 \frac{\partial^2}{\partial S^2} . \qquad (2)$$

(1) and (2) imply $S^2 \frac{\partial^2}{\partial S^2} = \frac{\partial^2}{\partial x^2} - \frac{\partial}{\partial x}$.

Thus

$$0 = -rf + rS\frac{\partial f}{\partial S} + rS^2 \frac{\partial^2 f}{\partial S^2} + \frac{\partial f}{\partial t}$$

$$= -rf + r\frac{\partial f}{\partial x} + r\left(\frac{\partial^2 f}{\partial x^2} - \frac{\partial f}{\partial x} \right) + \frac{\partial f}{\partial t}$$

$$= -rf + r\frac{\partial^2 f}{\partial x^2} + \frac{\partial f}{\partial t} .$$