**BIRKBECK**
**(University of London)**

**School of Computing and Mathematical Sciences**

# Financial Modelling and Data Science

**EMMS011S7**

**Homework 1**

Due via Moodle upload or email (to `b.baxter@bbk.ac.uk`) on January 22, 2024.

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Answer **ALL** questions. All questions carry the same weight.

You do **NOT** need to include detailed proofs of standard results from my notes unless the question explicitly asks for details.

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1. In this question, you may assume that $\mathbb{E}\exp(cZ) = \exp(c^2/2)$ when $Z \sim N(0,1)$, for any $c \in \mathbb{R}$.

   Let $S_t = e^{\sigma W_t}$, where $W_t$ is Brownian motion and $\sigma$ is a positive constant.

   (a) Show that
   $$\mathbb{E}\left(S_t S_T\right) = e^{\sigma^2(T+3t)/2},$$
   for $0 \le t \le T$.

   **5 pts**

   (b) Define the time average
   $$A_T = \frac{1}{T}\int_0^T S_t\,dt, \qquad \text{for } T > 0.$$
   Find $\mathbb{E}A_T$ and $\mathbb{E}(S_T A_T)$.

   **5 pts**

   (c) Let $W_{1,t}$ and $W_{2,t}$ be two independent Brownian motions. We define $W_{3,t}$ by
   $$W_{3,t} = W_{1,t}\cos\theta + W_{2,t}\sin\theta,$$
   where $0 < \theta < \pi/2$ is a constant. Find the matrix $M$, where
   $$\mathbb{E}(\mathbf{Z}_t\mathbf{Z}_t^T) = tM \qquad (*)$$
   and
   $$\mathbf{Z}_t = \begin{pmatrix} W_{1,t} \\ W_{3,t} \end{pmatrix}.$$

   **5 pts**

   (d) Find the symmetric positive definite matrix square-root
   $$M^{1/2} = \begin{pmatrix} a & b \\ b & a \end{pmatrix}$$
   where $M$ is given by (*).
   [**Hint:** You might find it useful to substitute $a = \cos\phi$ and $b = \sin\phi$, then find $\phi$ as a function of $\theta$.]

   **5pts**

2. (a) Let $X_t = \cos(W_t - c)$, where $c \in \mathbb{R}$ is a constant and $W_t$ denotes Brownian motion. Use Itô's lemma to find the SDE satisfied by $X_t$ and find $\mathbb{E}(\cos(W_t - c))$.

**6 pts**

(b) Let $f : \mathbb{R} \to \mathbb{R}$ be an infinitely differentiable function and let $W_t$ denote Brownian motion. Prove that

$$\frac{d}{dt}\mathbb{E}f(W_t) = \frac{1}{2}\mathbb{E}\left(f^{(2)}(W_t)\right).$$

Hence find $\mathbb{E}(W_t^4)$ and $\mathbb{E}(W_t^6)$.

**6 pts**

(c) Let $X \sim N(\mu, \sigma^2)$, where $\sigma$ is positive. Calculate the integral

$$\mathbb{E}\left(e^X - 1\right)_+.$$

[You may use standard properties of the cumulative distribution function

$$\Phi(x) = (2\pi)^{-1/2}\int_{-\infty}^{x} e^{-t^2/2}\,dt, \qquad x \in \mathbb{R},$$

without proof, including

$$\Phi(-x) = 1 - \Phi(x),$$

for $x \in \mathbb{R}$.]

**8 pts**

3. The drift-free Black–Scholes PDE is given by

$$0 = -rf + rS\frac{\partial f}{\partial S} + rS^2\frac{\partial^2 f}{\partial S^2} + \frac{\partial f}{\partial t}, \qquad (*)$$

where $r > 0$ is a constant.

(a) Prove that

$$f(S,t) = S^2 e^{3r(T-t)}$$

satisfies (*).

**5 points**

(b) Suppose $f(S,t) = V(t)$, i.e. $f$ depends *only* on time $t$. Solve (*) subject to the expiry condition $f(S_T, T) = K$, where $K$ is a constant.

**5 points**

(c) Hence, or otherwise, find the solution to (*) whose expiry price is given by

$$f(S_T, T) = S_T^2 - K.$$

**5 points**

(d) Use the transformation $S = e^x$ to restate (*) in terms of $x = \ln S$ instead of $S$.

**5 points**