Music and Sound

MS1: Quarter-comma meantone was the most common meantone temperament in the sixteenth and seventeeth centuries. We begin at D and ascend up and down in six **tempered** fifths: instead of using 3/2 we use $5^{1/4}$. In other words, we generate

$$A^\flat - E^\flat - B^\flat - F - C - G - \mathbf{D} - A - E - B - F^\sharp - C^\sharp - G^\sharp.$$

(a) Show that $C - E$ is a perfect third. Are there any other perfect thirds? [5]

(b) What is the difference between $A^\flat$ and $G^\sharp$ in this system? If the last step (i.e. $G^\sharp$) is replaced by a copy of $A^\flat$, but in the same octave as $G^\sharp$, find the interval $C^\sharp - G^\sharp$. Will the difference be audible? [5]

Solutions:

MS1 (a) Let $r = 5^{1/4}$, so that $r^4 = 5$ and $C - G - D - A - E$ takes us 2 octaves above, so the E immediately above C satisfies

$$\frac{P(E)}{P(C)} = 5/4,$$

where $P(E)$ denotes the pitch, or frequency, of E (this notation was used in class). There are 8 pure Major Thirds in this system:

$$A^\flat - C,\ C - E,\ E^\flat - G,\ G - B,\ B^\flat - D,\ D - F^\sharp,\ F - A,\ A - C^\sharp.$$

(b) If the frequency $P(D) = f$ Hz, then the meantone scale frequencies are as follows:

$A^\flat$	$E^\flat$	$B^\flat$	F	C	G	$\mathbf{D}$	A	E	B	$F^\sharp$	$C^\sharp$	$G^\sharp$
$r^{-6}f$	$r^{-5}f$	$r^{-4}f$	$r^{-3}f$	$r^{-2}f$	$r^{-1}f$	f	rf	r^2f	r^3f	r^4f	r^5f	r^6f

Thus

$$\frac{P(G^\sharp)}{P(A^\flat)} = r^{12} = 5^3 = 125$$

instead of $2^7 = 128$. The error in cents is

$$1200\frac{\ln\frac{128}{125}}{\ln 2} = 1200\left(\frac{7\ln 2 - 3\ln 5}{\ln 2}\right) \approx 41.06 \text{ cents}$$

and

$$\frac{128}{125} = \frac{2^7}{5^3} = \frac{2^{10}}{10^3} = 1.024.$$

If we set $G^\sharp$ equal to 7 octaves above $A^\flat$ to close the system, then

$$P(G^\sharp) = 2^7 r^{-6} f \quad \text{and} \quad P(C^\sharp) = r^5 f.$$

Hence

$$\frac{P(G^\sharp)}{P(C^\sharp)} = \frac{2^7 r^{-6}}{r^5} = \frac{2^7}{r^{11}} = \left(\frac{2^7}{r^{12}}\right) r = \frac{2^7}{5^3} r = \frac{2^{10}}{10^3} r = 1.024r \approx 1.53123715.$$

Comparing this to a perfect 5^{th}, we have the frequency ratio

$$W = \frac{\frac{2^{10}}{10^3} r}{\frac{3}{2}} = \frac{1.024r}{1.5}.$$

In cents, we find

$$1200 \frac{\ln W}{\ln 2} \approx 35.68 \text{ cents}$$

which is obviously audible, being much larger than 5 cents: this is the so-called *Wolf* interval of meantone you have heard in lectures.