COMMON PROBLEMS $M = QDQ^T$, $M^{\frac{1}{2}} = QD^{\frac{1}{2}}Q^T$

1. Find $M^{1/2}$ when $M = I + c\underline{u}\underline{u}^T$

$\|\underline{u}\| = 1$

SPECIAL CASE $\underline{u} = \underline{e}_1 = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$,

Then $M = I + c\underline{e}_1\underline{e}_1^T = \begin{pmatrix} 1+c & & & \\ & 1 & & \\ & & \ddots & \\ & & & 1 \end{pmatrix}$,

$$\underline{e}_1\underline{e}_1^T = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & & & \\ \vdots & & 0 & \\ 0 & & & \end{pmatrix},$$

EXERCISE: $M^{1/2} = \operatorname{diag}(\sqrt{1+c}, 1, \ldots, 1)$

$\underline{c > -1}$: $\underline{M^{\frac{1}{2}} = I + (\sqrt{1+c} - 1)\underline{e}_1\underline{e}_1^T}$.

$(Q^TQ = I)$

Now choose any orthogonal Q;

$$Q^T M Q = I + c\,Q^T\underline{e}_1\underline{e}_1^T Q$$

$$= I + c(Q^T\underline{e}_1)(Q^T\underline{e}_1)^T$$

$$= I + c\underline{y}\underline{y}^T.$$

Hence

$$Q^T M^{1/2} Q = I + (\sqrt{1+c} - 1)\underline{y}\underline{y}^T.$$

2014: 1d

$$\underline{e} = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$$

$$M = I + 20\,\underline{e}\,\underline{e}^T$$

$$M^{\frac{1}{2}} = I + c\,\underline{e}\,\underline{e}^T$$

So $$(M^{\frac{1}{2}})^2 = I + 2c\,\underline{e}\,\underline{e}^T + c^2\,\underline{e}\,\underbrace{\underline{e}^T\underline{e}}_{4}\,\underline{e}^T$$

$$= I + 2c\,\underline{e}\,\underline{e}^T + 4c^2\,\underline{e}\,\underline{e}^T.$$

Hence solve

$$20 = 2c + 4c^2$$

and pick positive root.

2. $M = \begin{pmatrix} A & B \\ B & A \end{pmatrix}$. CLAIM: $M^{\frac{1}{2}} = \begin{pmatrix} \alpha & \beta \\ \beta & \alpha \end{pmatrix}$.

EXERCISE: $M \begin{pmatrix} 1 \\ 1 \end{pmatrix} = (A+B) \begin{pmatrix} 1 \\ 1 \end{pmatrix}$,

$$M \begin{pmatrix} -1 \\ 1 \end{pmatrix} = (A-B) \begin{pmatrix} -1 \\ 1 \end{pmatrix}.$$

So $Q = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$, $Q^T Q = I$

and $M = Q D Q^T$, $D = \begin{pmatrix} A+B & 0 \\ 0 & A-B \end{pmatrix}$.

EXERCISE:

$$Q \begin{pmatrix} \lambda & 0 \\ 0 & \mu \end{pmatrix} Q^T = \begin{pmatrix} \frac{\lambda+\mu}{2} & \frac{\lambda-\mu}{2} \\ \frac{\lambda-\mu}{2} & \frac{\lambda+\mu}{2} \end{pmatrix}$$

i.e. $Q \begin{pmatrix} \lambda & 0 \\ 0 & \mu \end{pmatrix} Q^T = \begin{pmatrix} \gamma & \delta \\ \delta & \gamma \end{pmatrix}$.

Hence

$$M^{1/2} = Q \begin{pmatrix} \sqrt{\lambda} & 0 \\ 0 & \sqrt{\mu} \end{pmatrix} Q^T = \begin{pmatrix} \alpha & \beta \\ \beta & \alpha \end{pmatrix}.$$

2019 : 1d :

$$A = \begin{pmatrix} 7/4 & 3/4 & 3/4 & 3/4 \\ 3/4 & 7/4 & 3/4 & 3/4 \\ 3/4 & 3/4 & 7/4 & 3/4 \\ 3/4 & 3/4 & 3/4 & 7/4 \end{pmatrix}$$

$$A = I + \frac{3}{4} \underline{e}\,\underline{e}^T, \quad \underline{e} = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$$

Hence $A^{1/2} = I + c\, \underline{e}\,\underline{e}^T$.

Then

$$(A^{\frac{1}{2}})^2 = I + 2c\,\underline{e}\,\underline{e}^T + 4c^2\, \underline{e}\,\underline{e}^T$$

Hence

$$\frac{3}{4} = 2c + 4c^2$$

$$\left(c + \frac{1}{4}\right)^2 = \frac{1}{4}$$

$$c = 1/4 \quad \text{or} \quad c = -3/4.$$

2022 : 1d

$$M = \begin{pmatrix} 17 & 8 & 0 \\ 8 & 17 & 0 \\ 0 & 0 & 9 \end{pmatrix}$$

$$M^{\frac{1}{2}} = \begin{pmatrix} A^{1/2} & & 0 \\ & & 0 \\ 0 & 0 & 3 \end{pmatrix}$$

$$A^{1/2} = \begin{pmatrix} \alpha & \beta \\ \beta & \alpha \end{pmatrix}$$

$$\left(A^{\frac{1}{2}}\right)^2 = \begin{pmatrix} \alpha & \beta \\ \beta & \alpha \end{pmatrix}\begin{pmatrix} \alpha & \beta \\ \beta & \alpha \end{pmatrix}$$

$$= \begin{pmatrix} \alpha^2+\beta^2 & 2\alpha\beta \\ 2\alpha\beta & \alpha^2+\beta^2 \end{pmatrix}.$$

So $\alpha^2 + \beta^2 = 17$, $2\alpha\beta = 8$,

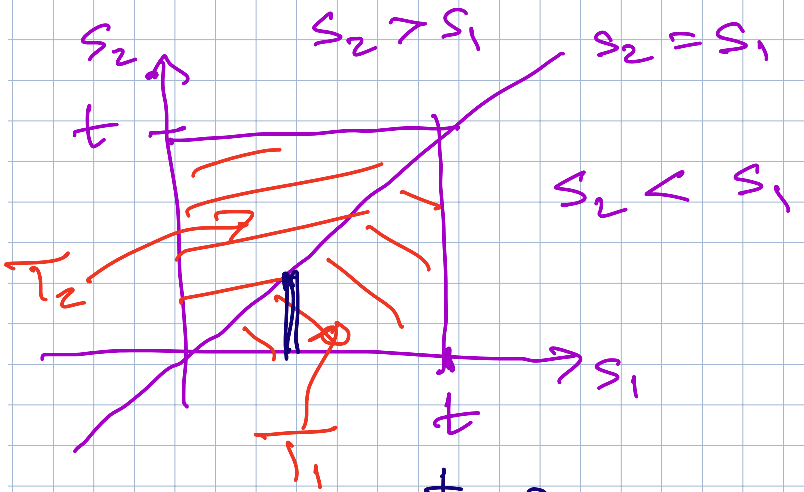
$$\alpha^2 + \left(\frac{4}{\alpha}\right)^2 = 17$$

2014:4 $\quad X_t = \int_0^t \underbrace{(2\pi)^{-1/2}\, e^{-s^2/2}}_{p(s)}\, W_s \, ds$

(b) $\mathbb{E}(X_t^2)$.

$$\mathbb{E}(X_t^2) = \mathbb{E}\int_0^t ds_1 \int_0^t ds_2 \; p(s_1)\, p(s_2)\, W_{s_1} W_{s_2}$$

$$= \int_0^t ds_1 \int_0^t ds_2 \; p(s_1)\, p(s_2)\, \underbrace{\mathbb{E}(W_{s_1} W_{s_2})}_{\min(s_1, s_2)}$$



$$\boxed{p(s_2)\, s_2 = -\frac{d}{ds_2}\, p(s_2)}$$

$$\mathbb{E}(X_t^2) = 2\int_0^t ds_1 \int_0^{s_1} ds_2 \; p(s_1)\, p(s_2)\, s_2$$

$$= 2\int_0^t ds_1 \, p(s_1) \underbrace{\int_0^{s_1} ds_2 \; p(s_2)\, s_2}_{\left[-p(s_2)\right]_{s_2=0}^{s_1}}$$

$$= 2\int_0^t ds_1 \, p(s_1)\left(1 - p(s_1)\right).$$

$$= 2 \underbrace{\int_0^t ds_1 \, p(s_1)}_{\Phi(t) - \frac{1}{2}} - 2 \underbrace{\int_0^t ds_1 \frac{1}{2\pi} e^{-s_1^2}}_{?}$$

TRICK: $s_1 = \dfrac{u_1}{\sqrt{2}}$

$ds_1 = \dfrac{du_1}{\sqrt{2}}$

2017: 7 NOT SET THIS YEAR

If you're stuck, let $d = 2$ first.

2019 : 2

$$0 = -rf + rS\frac{\partial f}{\partial S} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 f}{\partial S^2} + \frac{\partial f}{\partial t}$$

(a) $f(S,t) = S^2 e^{(r+\sigma^2)(T-t)}$

$= S^2 E$

$\frac{\partial f}{\partial S} = 2SE$, $\frac{\partial^2 f}{\partial S^2} = 2E$

$\frac{\partial f}{\partial t} = -(r+\sigma^2) S^2 E$.

So $S^2 E \big(\underbrace{-r + 2r + \sigma^2 - r - \sigma^2}_{=0} \big) = 0$

$= 0$.

(b) $f(S,t) = V(t)$

$\frac{\partial f}{\partial S} = \frac{\partial^2 f}{\partial S^2} = 0$

So $0 = -rV + \frac{dV}{dt}$

$V(t) = Ce^{rt}$.

But $f(S_T, T) = V(T) = K$

So $C e^{rT} = K$ or $C = K e^{-rT}$.

Then $V(t) = K e^{-rT} e^{rt}$

$= K e^{-r(T-t)}$

(c) $f(S_T, T) = S_T^2 - K$

(a): $f_1(S_t, t) = S^2 e^{(r+\sigma^2)(T-t)}$

So $f_1(S_T, T) = S_T^2$

(b) $f_2(S, t) = K e^{-r(T-t)}$

$f_2(S_T, T) = K$.

$f = f_1 - f_2$.

2022 : 4

(a) $f_k(S_T, T) = S_T^k$, $k \in \mathbb{Z}$

$$f_0(S_T, T) = 1$$

$$f_1(S_T, T) = S_T$$

$$f_2(S_T, T) = S_T^2$$

$$f_3(S_T, T) = S_T^3 \quad \ldots$$

$$f_k(S_0, 0) = e^{-rT} \sum_{l=0}^{n} \binom{n}{l} p^l (1-p)^{n-l} \times f_k(S_0 e^{l\alpha-(n-l)\alpha}, T)$$

$$= e^{-rT} \sum_{l=0}^{n} \binom{n}{l} p^l (1-p)^{n-l} S_0^k e^{kl\alpha} e^{-k(n-l)\alpha}$$

$$= S_0^k e^{-rT} \sum_{l=0}^{n} \binom{n}{l} (p e^{k\alpha})^l \left((1-p) e^{-k\alpha}\right)^{n-l}$$

$$= S_0^k e^{-rT} \left(p e^{k\alpha} + (1-p) e^{-k\alpha}\right)^n .$$

2022 : 6

$$\frac{\partial u}{\partial t} = \left(\frac{\sigma^2}{2}\right)\frac{\partial^2 u}{\partial x^2}$$

Explicit Euler:

$$\frac{\partial u}{\partial t} \simeq \frac{U_i^{n+1} - U_i^n}{k} \qquad k \equiv \Delta t$$

$$\frac{\partial^2 u}{\partial x^2} \simeq \frac{U_{i+1}^n - 2U_i^n + U_{i-1}^n}{h^2}$$

$h = \Delta x$

So EE:

$$\frac{U_i^{n+1} - U_i^n}{k} = \left(\frac{\sigma^2}{2}\right)\left(\frac{U_{i+1}^n - 2U_i^n + U_{i-1}^n}{h^2}\right)$$

OR $$U_i^{n+1} = U_i^n + \underbrace{\left(\frac{k}{h^2}\cdot\frac{\sigma^2}{2}\right)}_{\mu}\left(U_{i+1}^n - 2U_i^n + U_{i-1}^n\right)$$

STABILITY $\Longrightarrow$

$\mu \leq \frac{1}{2}$ NEED $\frac{k}{h^2} \leq \frac{2}{\sigma^2}\cdot\frac{1}{2} = \frac{1}{\sigma^2}$.

2022: 6b

$\cos w = (e^{iw} + e^{-iw})/2$

$\sin w = (e^{iw} - e^{-iw})/(2i)$

$$U_i^{n+1} = U_i^n + C\left(U_{i+2}^n - U_{i-2}^n + U_{i+2}^{n+1} - U_{i-2}^{n+1}\right)$$

$$v(x) = u(x) + C\begin{bmatrix} u(x+2h) - u(x-2h) \\ + v(x+2h) - v(x-2h) \end{bmatrix}$$

$$\hat{v}(z) = \hat{u}(z) + C\underbrace{(e^{2ihz} - e^{-2ihz})}_{2i\sin(2hz)}(\hat{u}(z) + \hat{v}(z))$$

$$\hat{v}(z)(1 - 2iC\sin(2hz)) = \hat{u}(z)(1 + 2iC\sin(2hz))$$

OK

$$\hat{v}(z) = \hat{u}(z)\underbrace{\left(\frac{1 + 2iC\sin(2hz)}{1 - 2iC\sin(2hz)}\right)}$$

$$\left|\frac{1+iP}{1-iP}\right| = \sqrt{\frac{1+P^2}{1+P^2}} = 1$$

REVISION: $W_t \sim N(0,t)$

$$\mathbb{E}\left(f(W_{t+h}, t+h) \mid W_t = X\right)$$

$$= \mathbb{E}\, f(\underbrace{W_{t+h} - W_t}_{N(0,h)} + \underbrace{W_t}_{X}, t+h) \mid W_t = X$$

$$= \mathbb{E}\, f(\sqrt{h}\, \underset{N(0,1)}{Z} + X, t+h)$$

$$= \int_{-\infty}^{\infty} (2\pi)^{-\frac{1}{2}} e^{-u^2/2} f(\sqrt{h}\,u + X, t+h)$$

EXAMPLE: $f(S,T) = \begin{cases} 1 & a \le S \le b \\ 0 & \text{else} \end{cases}$

Then

$$\mathbb{E}\left(f(W_T, T) \mid W_t = X\right)$$

$$= \int_{-\infty}^{\infty} (2\pi)^{-\frac{1}{2}} e^{-u^2/2} \underbrace{f(\sqrt{T-t}\,u + X, T)}\, du$$

$$\tau = T - t$$

$$= \begin{cases} 1 & \text{if } \sqrt{T-t}\,u + X \in (a,b) \\ 0 & \text{else} \end{cases}$$

$$= \begin{cases} 1 & \text{if } \frac{a-x}{\sqrt{\tau}} \leq u \leq \frac{b-x}{\sqrt{\tau}} \\ 0 \end{cases}$$

$$= \int_{\frac{a-x}{\sqrt{\tau}}}^{\frac{b-x}{\sqrt{\tau}}} (2\pi)^{-1/2} e^{-u^2/2} \, du$$

$$= \Phi\left(\frac{b-x}{\sqrt{\tau}}\right) - \Phi\left(\frac{a-x}{\sqrt{\tau}}\right).$$

BM: $\frac{d}{dt}\mathbb{E}(W_t^4)$

Let $X_t = W_t^4$.

Itô $\Longrightarrow$ $dX_t = 4W_t^3\,dW_t + 6W_t^2\,dt$

If $m(t) := \mathbb{E}X_t$, $dm(t) := \mathbb{E}\,dX_t$

and

$$dm(t) = 6\,\mathbb{E}(W_t^2)\,dt$$

i.e., $$\frac{dm}{dt} = 6\,\mathbb{E}(W_t^2)$$

OK $$\frac{d}{dt}\mathbb{E}(W_t^4) = 6\,\underbrace{\mathbb{E}(W_t^2)}_{t}.$$

cos $W_t \sim N(0,t)$

$$\frac{d}{dt}\mathbb{E}(W_t^4) = 6t$$

$$\mathbb{E}(W_t^4) = 3t^2 + \underbrace{C}_{=0} = 3t^2.$$

$Z \sim N(0,1)$

$\mathbb{E}(Z^4)$

$$= \int_{-\infty}^{\infty} s^4 (2\pi)^{-1/2} e^{-s^2/2} \, ds$$

$$= \int_{-\infty}^{\infty} s^3 (2\pi)^{-1/2} \underbrace{\left(s\, e^{-s^2/2}\right)}_{\frac{d}{ds}\left(-e^{-s^2/2}\right)} ds$$

$$= (2\pi)^{-\frac{1}{2}} \left\{ \left[s^3 \cdot \left(-e^{-s^2/2}\right) \right]_{-\infty}^{\infty} + \int_{-\infty}^{\infty} e^{-s^2/2} \cdot 3 s^2 \, ds \right\}$$

$$= 3 \underbrace{\int_{-\infty}^{\infty} s^2 (2\pi)^{-1/2} e^{-s^2/2} \, ds}_{\mathrm{Var}\, Z \,=\, 1}$$

$$= 3 .$$

2022 : 2

(a) $\mathbb{E}\, e^{iaW_t}$

$X_t = e^{iaW_t}$

Ito $\Rightarrow$ $dX_t = iaX_t\, dW_t - \frac{1}{2}a^2 X_t\, dt$.

$m(t) = \mathbb{E} X_t \Rightarrow$

$$\frac{dm}{dt} = -\frac{1}{2}a^2 m$$

$$m(t) = m(0)\, e^{-\frac{1}{2}a^2 t}.$$

$$\mathbb{E}\, e^{iaW_t} = e^{-\frac{1}{2}a^2 t}$$

(b) $e^{iaW_t} = \cos aW_t + i \sin aW_t$

$$e^{-\frac{1}{2}a^2 t} = \underbrace{\mathbb{E} \cos aW_t}_{e^{-\frac{1}{2}a^2 t}} + i \underbrace{\mathbb{E} \sin aW_t}_{0}$$

OR let $Y_t = \cos aW_t$.

Then $dY_t = -a \sin aW_t \, dW_t$

$- \frac{1}{2} a^2 \cos aW_t \, dt$

OR $\frac{dm}{dt} = -\frac{1}{2} a^2 m$

$\mathbb{E} \sin aW_t = e^{-\frac{t}{2}a^2} \big(\underbrace{\mathbb{E} \sin aW_0}_{=0} \big)$

(c) $X_t = \cosh aW_t = \frac{e^{aW_t} + e^{-aW_t}}{2}$

$X_t = e^{cW_t}$

$dX_t = c X_t \, dW_t + \frac{1}{2} c^2 X_t \, dt$

$m(t) = \mathbb{E} X_t \implies$

$\frac{dm}{dt} = \frac{1}{2} c^2 m$

$m(t) = e^{\frac{t}{2} c^2} m(0)$