BIRKBECK
(University of London)

MSc MATHEMATICAL FINANCE and sibling programmes

Department of Economics, Mathematics, and Statistics

Mathematical and Numerical Methods

Continuous Time Stochastic Processes

EMMS011S7

Homework 1

Due via Moodle upload or email (to `b.baxter@bbk.ac.uk`) on January 24, 2022.

Answer **ALL** questions. All questions carry the same weight.

You do **NOT** need to include detailed proofs of standard results from my notes unless the question explicitly asks for details.

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1. The three parts of this question are independent. You can assume standard results of stochastic calculus without proof if stated clearly.

   (a) Let W_t denote univariate Brownian motion and let

   $$S_t = e^{a+bt+\sigma W_t}, \qquad \text{for } t \geq 0,$$

   where a, b and σ are constants. Find conditions on a, b and σ for which the discounted process $X_t = \exp(-rt)S_t$ is a martingale, where r is the constant risk-free interest rate.

   (b) The drift-free Black–Scholes PDE is given by

   $$0 = -rf + rS\frac{\partial f}{\partial S} + rS^2\frac{\partial^2 f}{\partial S^2} + \frac{\partial f}{\partial t}, \qquad (*)$$

   where r is a constant. Find the constant q for which

   $$f(S,t) = S^3 e^{qt}$$

   satisfies (*). Further, find the general solution to (*) when $f(S,t)$ does not depend explicitly on S and has expiry value M at expiry time T, that is, $f(S,t) = g(t)$ for some function $g(t)$.

   (c) The general Black–Scholes PDE is given by

   $$0 = -rf + rS\frac{\partial f}{\partial S} + \frac{1}{2}\sigma^2 S^2\frac{\partial^2 f}{\partial S^2} + \frac{\partial f}{\partial t},$$

   where r and σ are constants. Find the resulting PDE if we substitute $x = \ln S$.

2. The four parts of this question are independent. You can assume standard results of stochastic calculus without proof if stated clearly.

   (a) Let W_t be a Brownian motion. Let $X_t = \exp(-izW_t)$, where $z \in \mathbb{R}$ and $i = \sqrt{-1}$, and find the SDE satisfied by X_t. Hence calculate $\mathbb{E}X_t$.

   (b) Let $W_{1,t}$, $W_{2,t}$ and $W_{3,t}$ be three independent Brownian motions and define the three-dimensional Brownian motion $\mathbf{Z}_t$ by

$$\mathbf{Z}_t = \begin{pmatrix} 2 & 0 & 0 \\ 1/2 & \sqrt{15/4} & 0 \\ 0 & \sqrt{4/15} & \sqrt{56/15} \end{pmatrix} \begin{pmatrix} W_{1,t} \\ W_{2,t} \\ W_{3,t} \end{pmatrix}.$$

   Find $\mathbb{E}\left(\mathbf{Z}_t\mathbf{Z}_t^T\right)$.

   (c) Let $g(x) = p + qx + \frac{1}{2}rx^2$, for $x \in \mathbb{R}$, where p, q and r are constants. Find dX_t when $X_t = g(W_t)$ and W_t denotes univariate Brownian motion.

   (d) Let $W_{1,t}$ and $W_{2,t}$ be correlated univariate Brownian motions that satisfy $dW_{1,t}dW_{2,t} = \rho\, dt$, where $-1 < \rho < 1$ is a constant. Find the infinitesimal increment dY_t when $Y_t = f(W_{1,t}, W_{2,t})$ and $f(x_1, x_2)$ is the quadratic

$$f(x_1, x_2) = \frac{1}{2}\left(Ax_1^2 + 2Bx_1x_2 + Cx_2^2\right),$$

   where A, B and C are constants.

3. Let $W_{1,t}$ and $W_{2,t}$ be correlated Brownian motions satisfying $dW_{1,t}dW_{2,t} = \rho\, dt$, where $\rho \in [-1, 1]$ is a constant. Define

$$Y_t = e^{\sigma_1 W_{1,t} + \sigma_2 W_{2,t}}, \qquad \text{for } t \geq 0.$$

(a) Prove that

$$\frac{dY_t}{Y_t} = \sigma_1 dW_{1,t} + \sigma_2 dW_{2,t} + \frac{1}{2}\left(\sigma_1^2 + 2\sigma_1\sigma_2\rho + \sigma_2^2\right) dt. \qquad (*)$$

(b) Let $m(t) = \mathbb{E}Y_t$ and use (*) to derive, and solve, the differential equation satisfied by $m(t)$.

(c) Define

$$S_i(t) = S_i(0)e^{(r-\sigma_i^2/2)t+\sigma_i W_{i,t}}, \qquad i = 1, 2.$$

Hence, or otherwise, use the solution $m(t)$ to derive the correlation coefficient for $S_1(t)$ and $S_2(t)$.

4. The Ornstein–Uhlenbeck stochastic differential equation is given by

$$dX_t = \alpha\left(\theta - X_t\right)dt + \sigma dW_t, \qquad (*)$$

where α and θ are real constants and W_t denotes Brownian motion.

(a) Let $\widehat{X_t}$ denote the solution to (*) when $\sigma = 0$. Find $\widehat{X_t}$ and calculate the limit $\lim_{t\to\infty}\widehat{X_t}$.

(b) Let $Y_t = X_t - \theta$. Prove that

$$d\left(Y_t e^{\alpha t}\right) = \sigma e^{\alpha t} dW_t.$$

(c) Hence, or otherwise, prove that the solution to (*) is given by

$$X_t = \widehat{X_t} + \sigma e^{-\alpha t}\int_0^t e^{\alpha s}\, dW_s. \qquad (**)$$

(d) Show that the solution X_t given by (**) satisfies

$$\text{var } X_t = \sigma^2\left(\frac{1 - e^{-2\alpha t}}{2\alpha}\right).$$

[You may use any relevant theorem of stochastic integration without proof if clearly stated.]