BIRKBECK
(University of London)

MSc MATHEMATICAL FINANCE and sibling programmes

Department of Economics, Mathematics, and Statistics

Mathematical and Numerical Methods

EMMS011S7

Homework 2

Due via Moodle upload or email (to `b.baxter@bbk.ac.uk`) on April 25, 2022.

Answer **ALL** questions. All questions carry the same weight.

You do **NOT** need to include detailed proofs of standard results from my notes unless the question explicitly asks for details.

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1. (a) Calculate the infinitesimal increment for $X_t = W_t^2$. Hence, or otherwise, prove that

$$\int_0^u W_t dW_t = \frac{1}{2}\left(W_u^2 - u\right).$$

(b) Prove that

$$h(t)W_t = \int_0^t h(s)dW_s + \int_0^t h'(s)W_s ds$$

for any infinitely differentiable function $h(t)$.

(c) Define

$$D_n(t) = \sum_{k=0}^{n-2} \left(\Delta_0^2 W_{kh}\right)^2, \qquad \text{for } t > 0,$$

where $n \geq 2$ is a positive integer, $h = t/n$ and

$$\Delta_0^2 f(x) = f(x+h) - 2f(x) + f(x-h)$$

is the second order central difference. Find $\mathbb{E}D_n(t)$ and $\lim_{n\to\infty} \mathbb{E}D_n(t)$.

(d) Find the eigenvalues of the covariance matrix for the multivariate Brownian motion in Question 1(b).

2. In this question, you may use the fact that the Fourier transform of the Gaussian probability density function

$$G_{\sigma^2}(\mathbf{x}) = (2\pi\sigma^2)^{-d/2} e^{-\|\mathbf{x}\|^2/(2\sigma^2)}, \qquad \text{for } \mathbf{x} \in \mathbb{R}^d,$$

is given by

$$\widehat{G_{\sigma^2}}(\mathbf{z}) = e^{-\sigma^2\|\mathbf{z}\|^2/2},$$

for $\mathbf{z} \in \mathbb{R}^d$ and $\sigma > 0$.

(a) Let $u(x, t)$ satisfy the $1D$ time-reversed diffusion equation

$$\frac{\partial u}{\partial t} + \frac{1}{2}\alpha^2 \frac{\partial^2 u}{\partial x^2} = 0 \qquad (*)$$

and define

$$\widehat{u}(z, t) = \int_{-\infty}^{\infty} u(x, t) e^{-ixz}\, dx, \qquad z \in \mathbb{R}.$$

Use the Fourier transform to construct the ordinary differential equation satisfied by $\widehat{u}(z, t)$. Hence solve (*) when $u(x, T) = G_{\sigma_1^2}(x + 1) - G_{\sigma_2^2}(x - 1)$.

(b) The 9-point difference approximation to the Laplacian is given by

$$\begin{aligned} L_h f(x, y) & \\ := \ & \frac{1}{3h^2}\Big[-10 f(x, y) \\ & +2\Big(f(x + h, y) + f(x - h, y) + f(x, y + h) + f(x, y - h)\Big) \\ & +\frac{1}{2}\Big(f(x + h, y + h) + f(x + h, y - h) + f(x - h, y + h) + f(x - h, y - h)\Big)\Big]. \end{aligned}$$

Use the Fourier transform to prove that

$$L_h f(x, y) = \nabla^2 f(x, y) + \frac{h^2}{12}\nabla^4 f(x, y) + O(h^4).$$

3. (a) Consider the explicit Euler scheme

$$U_i^{n+1} = U_i^n + \mu\left(U_{i+1}^n - 2U_i^n + U_{i-1}^n\right), \qquad 1 \le i \le M-1, \quad n \ge 0,$$

where $\mu = \Delta t/(\Delta x)^2$ and $\{U_i^n : 0 < i < M\}$, $\{U_0^n : n \ge 0\}$, $\{U_M^n : n \ge 0\}$ are given. Find the local truncation error of the scheme when (i) $\mu \neq 1/6$, and (ii) $\mu = 1/6$.

(b) Use the von Neumann stability criterion to prove that the scheme

$$U_m^{n+1} - \frac{4}{3}U_m^n + \frac{1}{3}U_m^{n-1} = \frac{9}{2}\mu\left(U_{m+1}^{n+1} - 2U_m^{n+1} + U_{m-1}^{n+1}\right)$$

is stable for all positive μ.