10. $y'(t) = -10^4\left(y - \frac{1}{t}\right) - \frac{1}{t^2}, \quad t \geq 1, \quad y(1) = 1$

Analytic solution $y(t) = \frac{1}{t}, \quad t \geq 1.$

Then

$$y_{n+1} = y_n + h_n\left[-10^4\left(y_n - \frac{1}{t_n}\right) - \frac{1}{t_n^2}\right]. \qquad (1)$$

CLAIM: $y_{n+1} - y(t_{n+1})$

$$= (1 - 10^4 h_n)(y - y(t_n)) - \frac{h_n^2}{t_n^2 t_{n+1}}. \qquad (2)$$

Proof: Knowing the analytic solution, we find the LTE

$$y(t_{n+1}) - y(t_n) - h_n\left(-10^4\underbrace{\left(y(t_n) - \frac{1}{t_n}\right)}_{=0} - \frac{1}{t_n^2}\right)$$

$$= \frac{1}{t_{n+1}} - \frac{1}{t_n} + \frac{h_n}{t_n^2}$$

$$= \frac{h_n^2}{t_n^2 t_{n+1}}. \qquad (3)$$

Subtract (3) from (1) to obtain (2). $\square$

If $h_n = 2 \times 10^{-4}$ and $e_n = y_n - y(t_n)$, then $10^4 h_n = 2$ and

$$e_{n+1} = -e_n - \frac{4 \times 10^{-8}}{t_n^2 t_{n+1}}. \qquad (4)$$

Key idea: If $n = O(10^4)$ then $t_n = O(1)$ and $|\text{RHS}| > 10^{-6}$, hence failure.