7. If $y' = y$ then

$$k_1 = y_n$$ $k_1$

$$k_2 = y_n + \tfrac{1}{3}hk_1 = (1 + \tfrac{1}{3}h)y_n$$ $k_2$

$$k_3 = y_n - \tfrac{1}{3}hk_1 + hk_2$$
$$= y_n - \tfrac{1}{3}hy_n + h(1 + \tfrac{1}{3}h)y_n$$

or $$k_3 = \left(1 + \tfrac{2}{3}h + \tfrac{1}{3}h^2\right)y_n$$ $k_3$

and

$$k_4 = y_n + hk_1 - hk_2 + hk_3$$
$$= y_n + hy_n - h(1 + \tfrac{1}{3}h)y_n$$
$$+ h(1 + \tfrac{2}{3}h + \tfrac{1}{3}h^2)y_n,$$

i.e. $$k_4 = \left(1 + h + \tfrac{1}{3}h^2 + \tfrac{1}{3}h^3\right)y_n .$$ $k_4$

Hence

$$y_{n+1} = y_n + \tfrac{1}{8}h(k_1 + 3k_2 + 3k_3 + k_4)$$

$$= \left\{ 1 + \tfrac{1}{8}h\left( 1 + 3(1 + \tfrac{1}{3}h) + 3(1 + \tfrac{2}{3}h + \tfrac{1}{3}h^2) + 1 + h + \tfrac{1}{3}h^2 + \tfrac{1}{3}h^3 \right) \right\} y_n$$

$$= \left\{ 1 + h + \frac{h^2}{8}(1 + 2 + 1) + \frac{h^3}{8}(1 + \tfrac{1}{3}) + \frac{h^4}{8}(\tfrac{1}{3}) \right\} y_n$$

$$= \left\{ 1 + h + \tfrac{1}{2}h^2 + \tfrac{1}{6}h^3 + \tfrac{1}{24}h^4 \right\} y_n .$$

If $f$ is independent of $y$, then

$$y_{n+1} = y_n + \frac{1}{8}h\left(f(t_n) + 3f(t_n+\frac{h}{3}) + 3f(t_n+\frac{2h}{3}) + f(t_n+h)\right)$$

i.e. we need the error in

$$L(y) = y(1) - y(0) - \frac{1}{8}\left(y'(0) + 3y'(\tfrac{1}{3}) + 3y'(\tfrac{2}{3}) + y'(1)\right)$$

$L(1) = 0$

| $y$ | $y(1)$ | $y(0)$ | $\frac{1}{8}\left(y'(0)+3y'(\frac{1}{3})+3y'(\frac{2}{3})+y'(1)\right)$ | $L(y)$ |
|---|---|---|---|---|
| $x$ | 1 | 0 | $\frac{1}{8}\cdot 8 = 1$ | 0 |
| $x^2$ | 1 | 0 | $\frac{2}{8}(3\cdot\frac{1}{3}+3\cdot\frac{2}{3}+1) = 1$ | 0 |
| $x^3$ | 1 | 0 | $\frac{3}{8}\big(\underbrace{3\cdot\frac{1}{3^2}+3\cdot\frac{4}{3^2}+1}_{\frac{1}{3}+\frac{4}{3}+1=\frac{8}{3}}\big) = 1$ | 0 |
| $x^4$ | 1 | 0 | $\frac{1}{2}\big(\underbrace{3\cdot\frac{1}{3^3}+3\cdot\frac{2^3}{3^3}+1}_{\frac{1}{9}+\frac{8}{9}+1}\big) = 1$ | 0 |
| $x^5$ | 1 | 0 | $\frac{5}{8}\big(\underbrace{3\cdot\frac{1}{3^4}+3\cdot\frac{2^4}{3^4}+1}_{\frac{1}{27}+\frac{48}{27}+\frac{27}{27}}\big) \neq 1$ | |