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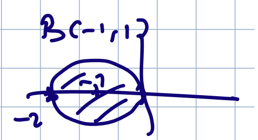
(i) $y_{n+1} = y_n + h f(t_n, y_n)$

If $y' = \lambda y$, then $y_{n+1} = (1 + h\lambda) y_n$



and we need $z = h\lambda$ s.t. $|1 + z| < 1$, i.e.

$$D = \{ z \in \mathbb{C} : |1 + z| < 1 \} = B(-1, 1).$$

and $D \cap \mathbb{R} = (-2, 0)$.

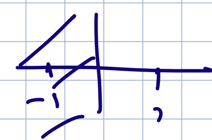
(ii) $y_{n+1} = y_n + \frac{1}{2} h (\lambda y_n + \lambda y_{n+1})$

or $\left(1 - h\frac{\lambda}{2}\right) y_{n+1} = \left(1 + \frac{h\lambda}{2}\right) y_n$

i.e.

$$y_{n+1} = \left(\frac{1 + \frac{h\lambda}{2}}{1 - \frac{h\lambda}{2}} \right) y_n$$

and $D = \left\{ z \in \mathbb{C} : \left| \frac{1 + z/2}{1 - z/2} \right| < 1 \right\}$



$= \{ z \in \mathbb{C} : \frac{z}{2}$ nearer to -1 than $1 \}$

i.e. $D = \{ z \in \mathbb{C} : \mathrm{Re}\, z < 0 \}$,

so $D \cap \mathbb{R} = (-\infty, 0)$.

(iii) $y_{n+2} = y_n + 2h\lambda y_{n+1}$

or $(z = h\lambda)$ $y_{n+2} - 2z y_{n+1} - 1 = 0$.

Now $(\mu - \mu_1)(\mu - \mu_2) = \mu^2 - 2\mu z - 1 = 0$

$\Rightarrow \mu_1 \mu_2 = -1$ and $\mu = z \pm \sqrt{(z^2 + 1)}$.

These roots are real for $z \in \mathbb{R}$ and at least one has modulus > 1 $(|\mu_1| \cdot |\mu_2| = 1)$ unless $z = 0$, so $D \cap \mathbb{R} = \{0\}$.

(iv) $$y_{n+2} = y_{n+1} + h\lambda\left(\frac{3}{2} y_{n+1} - \frac{1}{2} y_n\right)$$

or $(z = h\lambda)$

$$y_{n+2} - \left(1 + \frac{3}{2} z\right) y_{n+1} + \frac{1}{2} z y_n = 0.$$

Therefore we consider the associated quadratic

$$\mu^2 - \left(1 + \frac{3}{2} z\right)\mu + \frac{1}{2} z = 0.$$

The roots are real for $z \in \mathbb{R}$ since

$$\begin{aligned}
\text{"}b^2 - 4ac\text{"} &= \left(1 + \frac{3}{2} z\right)^2 - 2z \\
&= 1 + 3z + \frac{9}{4} z^2 - 2z \\
&= \frac{9}{4} z^2 + z + 1 \\
&= \frac{9}{4}\left(z^2 + 2 \cdot \frac{2}{9} z + \frac{4}{9}\right) \\
&= \frac{9}{4}\left(\left(z + \frac{2}{9}\right)^2 + \frac{4}{9}\left(1 - \frac{1}{9}\right)\right) \\
&= \frac{9}{4}\left(\left(z + \frac{2}{9}\right)^2 + \frac{4}{9} \cdot \frac{8}{9}\right) \\
&= \frac{9}{4}\left(z + \frac{2}{9}\right)^2 + \frac{8}{9} \geqslant 0 \quad \text{for } z \in \mathbb{R}.
\end{aligned}$$

Therefore let $\mu = \pm 1$ in $\mu^2 - \left(1 + \frac{3}{2} z\right)\mu + \frac{1}{2} z = 0$:

$\mu = 1 \Rightarrow 0 = 1 - \left(1 + \frac{3}{2} z\right) + \frac{1}{2} z = -z$ or $z = 0$.

$\mu = -1 \Rightarrow 0 = 1 + \left(1 + \frac{3}{2}\right) z + \frac{1}{2} z = 2 + 2z$ or $z = -1$.

Hence $D \cap \mathbb{R} = (-1, 0)$.

(v) $y' = \lambda y \Rightarrow$ $k_1 = \lambda y_n,$

$$k_2 = \lambda(y_n + hk_1)$$
$$= \lambda y_n + h\lambda^2 y_n$$

i.e. $k_2 = (\lambda + h\lambda^2)\, y_n.$

Then

$$y_{n+1} = y_n + \tfrac{1}{2}h(k_1 + k_2)$$
$$= y_n\left(1 + \tfrac{1}{2}h\lambda + \tfrac{1}{2}h\lambda + \tfrac{1}{2}h^2\lambda^2\right)$$
$$= y_n(1 + h\lambda + \tfrac{1}{2}h^2\lambda^2).$$

Thus $D = \{ z \in \mathbb{C} : |1 + z + \tfrac{1}{2}z^2| < 1 \}.$

Now $1 + z + \tfrac{1}{2}z^2 = 1 \Rightarrow z(1 + \tfrac{z}{2}) = 0 \Rightarrow z = 0$ or -2.

If $1 + z + \tfrac{1}{2}z^2 = -1,$

then $\tfrac{1}{2}z^2 + z + 2 = 0$

or $z^2 + 2z + 4 = 0$

i.e. $(z+1)^2 + 3 = 0,$

so no real solutions; Hence $D \cap \mathbb{R} = (-2, 0).$