We let $\quad L(y) = y(h) - y(0) - \frac{h}{2}\left(y'(0) + y'(h)\right).$

EXERCISE: Show that $L(y) = 0$ for $y(x) = 1, x, x^2$, but

$$L(x \mapsto x^3) = -\tfrac{1}{2} h^3. \qquad (1)$$

The PKT implies that

$$L(y) = \int_0^h K(t)\, y^{(3)}(t)\, dt$$

where

$$K(t) = \tfrac{1}{2}\, L\left(x \longmapsto (x-t)_+^2\right)$$

or

$$K(t) = \tfrac{1}{2}\left\{(h-t)_+^2 - (0-t)_+^2 - \frac{h}{2}\left(2(h-t)_+ + 2(0-t)_+\right)\right\}.$$

Now $K(t) = 0$ for $t \notin [0, h]$ and, for $0 \le t \le h$,

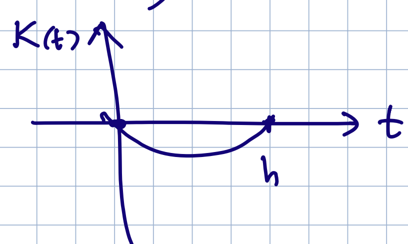
$$K(t) = \tfrac{1}{2}\left\{(h-t)^2 - h(h-t)\right\}$$

$$\cancel{h^2} - 2ht + t^2$$
$$-\cancel{h^2} + ht = t^2 - ht$$

$$= \tfrac{1}{2}(t^2 - ht)$$

$$= \tfrac{1}{2}\, t(t-h).$$

Thus $K(t) < 0$ for $0 < t < h$.



We can easily calculate $\int_0^h K$ BUT it's easier to let

$$y(t) = t^3$$

so that $y^{(3)}(t) \equiv 6$ and

$$L(x \mapsto x^3) = 6 \int_0^h K,$$

Using (1),

$$\int_0^h K(t)\,dt = -\frac{1}{12}$$

and

$$|L(y)| \leq \frac{1}{12} \|y^{(3)}\|_\infty$$

with equality when $y(x) = x^3$.

TAYLOR SERIES ROUTE: we have

$$y\left(\tfrac{h}{2}\right) = y(0) + \tfrac{h}{2} y'(0) + \tfrac{1}{2}\left(\tfrac{h}{2}\right)^2 y''(0) + O\left(h^3 \|y^{(3)}\|_\infty\right)$$

AND

$$y\left(\tfrac{h}{2}\right) = y(h) - \tfrac{h}{2} y'(h) + \tfrac{1}{2}\left(\tfrac{h}{2}\right)^2 y''(h) + O\left(h^3 \|y^{(3)}\|_\infty\right).$$

Subtracting these equations:

$$0 = y(h) - y(0) - \tfrac{h}{2}\left(y'(h) + y'(0)\right)$$

$$+ \tfrac{1}{8} h^2 \underbrace{\left(y''(h) - y''(0)\right)}_{= h\, y'''(\alpha),\ \alpha \in (0,h)\ (\text{MVT})} + O\left(h^3 \|y^{(3)}\|_\infty\right)$$

or $y(h) - y(0) - \frac{h}{2}\left(y'(h) + y'(0)\right) = O\left(h^3 \|y^{(3)}\|_\infty\right).$