9. Let

$$\Delta^2 u(x,t) = u(x+h,t) - 2u(x,t) + u(x-h,t).$$

Then the local truncation error is

$$L(x,t) = \left(1 - \tfrac{1}{2}(\mu-\theta)\Delta^2\right) u(x, t+k)$$
$$- \left[1 + \tfrac{1}{2}(\mu+\theta)\Delta^2\right] u(x,t) \qquad (1)$$

where $k = \mu h^2$ and $u_t = u_{xx}$.

DON'T Taylor expand (1) directly: pages of algebra!

TRICK:

$$u(x, t+k) = \sum_{l=0}^{\infty} \frac{k^l}{l!} \partial_t^l u(x,t)$$

where $\partial_t \equiv \dfrac{\partial}{\partial t}$, $\partial_t^l \equiv \dfrac{\partial^l}{\partial t^l}$.

BUT $\partial_t u = D^2 u$ where $D \equiv \dfrac{\partial}{\partial x}$.

AND $k = \mu h^2$.

SO

$$u(x, t+k) = \sum_{l=0}^{\infty} \frac{\mu^l h^{2l}}{l!} D^{2l} u(x,t)$$

$$= \left(1 + \mu h^2 D^2 + \tfrac{1}{2}\mu^2 h^4 D^4 + \cdots\right) u(x,t).$$

FURTHER

$$\Delta^2 u(x,t) = \left(h^2 D^2 + \frac{1}{12} h^4 D^4 + \cdots\right) u(x,t).$$

HENCE $L(x,t) = \hat{L}\, u(x,t)$ where

$\hat{L} =$

$$\left(1 + \mu h^2 D^2 + \frac{1}{2}\mu^2 h^4 D^4 + \cdots\right)\left[1 - (\mu - \theta)\left(\frac{1}{2} h^2 D^2 + \frac{1}{4!} h^4 D^4 + \cdots\right)\right]$$

$$- \left[1 + (\mu + \theta)\left(\frac{1}{2} h^2 D^2 + \frac{1}{4!} h^4 D^4 + \cdots\right)\right]$$

$$= h^2 D^2 \left(\underbrace{\mu - \frac{1}{2}(\mu - \theta) - \frac{1}{2}(\mu + \theta)}_{=0}\right)$$

$$+ h^4 D^4 \left\{\underbrace{\frac{1}{2}\mu^2 - \frac{1}{2}\mu(\mu - \theta) - \frac{1}{4!}(\mu - \theta) - \frac{1}{4!}(\mu + \theta)}_{\text{EXERCISE : SIMPLIFY}}\right\}$$

$$+ O\left(h^6 D^6\right).$$