36. HINTS FOR SOLUTION

1. $A^2\underline{x} = c_0\underline{x} + c_1 A\underline{x}$. FIND c_0, c_1.

Choose a new basis $\underline{m}_1, \underline{m}_2, \underline{m}_3, \underline{m}_4$ for $\mathbb{R}^4$ where $\underline{m}_1 = \underline{x}$, $\underline{m}_2 = A\underline{x}$. The matrix of A w.r.t. the new basis is

$$A_{new} = \left(\begin{array}{cc|c} 0 & c_0 & N \\ 1 & c_1 & \\ \hline \multicolumn{2}{c|}{O} & \hat{A} \end{array}\right) \leftarrow \text{DEFLATION OF } A.$$

and

$$A_{new} = M^{-1}AM \quad \text{where} \quad M = (\underline{m}_1\ \underline{m}_2\ \underline{m}_3\ \underline{m}_4)$$

Then the eigenvalues of A are the eigenvalues of the 2×2 matrices $\begin{pmatrix} 0 & c_0 \\ 1 & c_1 \end{pmatrix}$ and $\hat{A}$.

2. Finding $\hat{A}$: Let $\underline{m}_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}$, $\underline{m}_4 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$

so that

$$M = \begin{pmatrix} P & O \\ Q & I \end{pmatrix}, \qquad P, Q \ 2\times 2 \text{ matrices.}$$

EXERCISE: Show that

$$M^{-1} = \begin{pmatrix} R & O \\ S & I \end{pmatrix}$$

where $R = P^{-1}$ and $S = -QP^{-1}$!

3. Let $A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}$

where $A_{11}, A_{12}, A_{21}, A_{22}$ are 2×2 submatrices of A.

EXERCISE : Show that

$$A_{new} = M^{-1} A M = \left(\begin{array}{cc|c} 0 & c_0 & \\ 1 & c_1 & N \\ \hline \multicolumn{2}{c|}{O} & \underbrace{-QP^{-1}A_{12} + A_{22}}_{\hat{A}} \end{array}\right).$$

Find the eigenvalues of the 2×2 matrix $\hat{A}$.