2022/2/17

17C

A scalar, autonomous, ordinary differential equation $y' = f(y)$ is solved using the Runge–Kutta method

$$\begin{aligned} k_1 &= f(y_n)\,, \\ k_2 &= f(y_n + (1-a)hk_1 + ahk_2)\,, \\ y_{n+1} &= y_n + \frac{h}{2}(k_1 + k_2)\,, \end{aligned}$$

where h is a step size and a is a real parameter.

(a) Determine the order of the method and its dependence on a.

(b) Find the range of values of a for which the method is A-stable.

(a) Taylor for $y' = f$, $y'' = f f_y$,

$$y''' = f\partial_y(f f_y) = f^2 f_{yy} + f f_y^2,$$

i.e.

$$y_{n+1} = y_n + hf + \tfrac{1}{2}h^2 f f_y + \tfrac{1}{6}h^3 (f f_y^2 + f^2 f_{yy}) + O(h^4). \quad (1)$$

Stop at h^3 term : after is horrible $\Rightarrow$ better method needed!

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$$k_2 = f(y_n + (1-a)hf + ahk_2). \quad (2)$$

Evaluate RHS using Taylor :

$$f(y_n + h[(1-a)f + ak_2])$$

$$= f + hf_y[(1-a)f + ak_2] + \tfrac{1}{2}h^2 f_{yy}[(1-a)f + ak_2]^2 + O(h^3) \quad (3)$$

Then use this for fixed point iteration :

$$k_2^{new} = f(y_n + h[(1-a)f + ak_2^{old}]). \quad (4)$$

k_2^{old} : Let's start with $k_2^{old} = f$, so (4) implies $k_2^{new} = f(y_n + hf)$.

So now we let

$$k_2^{old} = f + hff_y + \tfrac{1}{2}h^2 f^2 f_{yy} + O(h^3).$$

Substituting in 4 :

$$k_2^{new} = f\left(y_n + h(1-a)f + ha\left[f + hff_y + \tfrac{1}{2}h^2f^2f_{yy}\right]\right)$$

$$= f\left(y_n + hf + ah^2ff_y + \tfrac{1}{2}ah^3f^2f_{yy} + O(h^4)\right)$$

$$= f + f_y\left(hf + ah^2ff_y + \tfrac{1}{2}ah^3f^2f_{yy} + O(h^4)\right)$$
$$+ \tfrac{1}{2}f_{yy}\left(hf + ah^2ff_y + \tfrac{1}{2}ah^3f^2f_{yy}\right)^2$$

or

$$k_2^{new} = f + hff_y + h^2\left(aff_y^2 + \tfrac{1}{2}f_{yy}f^2\right) + O(h^3).$$

So

$$y_{n+1} = y_n + \tfrac{1}{2}h(k_1 + k_2)$$
$$= y_n + \tfrac{1}{2}hf + \tfrac{1}{2}h\left[f + hff_y + h^2\left(aff_y^2 + \tfrac{1}{2}f_{yy}f^2\right) + O(h^3)\right]$$

OR

$$y_{n+1} = y_n + hf + \tfrac{1}{2}h^2ff_y$$
$$+ \tfrac{1}{2}h^3\left[aff_y^2 + \tfrac{1}{2}f_{yy}f^2\right] + O(h^4)$$

But Taylor $\Rightarrow$

$$y_{n+1} = y_n + hf + \tfrac{1}{2}h^2ff_y$$
$$+ \tfrac{1}{6}h^3\left[ff_y^2 + f_{yy}f^2\right] + O(h^4)$$

$\Rightarrow O(h^3)$ error.

(b) If $y' = \lambda y$, $\lambda < 0$, then

$$k_1 = \lambda y_n,$$
$$k_2 = \lambda[y_n + (1-a)h\lambda y_n + ahk_2]$$

or $(1 - ah\lambda)k_2 = \lambda[1 + (1-a)h\lambda]y_n$,

i.e. $$k_2 = \frac{\lambda[1 + (1-a)h\lambda]y_n}{1 - ah\lambda}.$$

If we set $z = h\lambda$, then

$$y_{n+1} = y_n + \frac{h}{2}(k_1 + k_2)$$

$$= y_n + \frac{h\lambda}{2}y_n + \frac{h\lambda}{2}\frac{[1 + (1-a)h\lambda]y_n}{1 - ah\lambda}$$

$$= y_n\left\{1 + \frac{1}{2}z + \frac{1}{2}z\frac{[1 + (1-a)z]}{1 - az}\right\}$$

$$= y_n\left\{\frac{(1 + \frac{1}{2}z)(1 - az) + \frac{1}{2}z[1 + (1-a)z]}{1 - az}\right\}$$

$$\text{Numerator} = 1 + (\tfrac{1}{2} - a)z - \tfrac{1}{2}az^2 + \tfrac{1}{2}z + \tfrac{1}{2}(1-a)z^2$$

$$= 1 + (1-a)z + (\tfrac{1}{2} - a)z^2$$

so

$$y_{n+1} = y_n \underbrace{\left[\frac{1 + (1-a)z + (\frac{1}{2} - a) z^2}{1 - az} \right]}_{R(z).}$$

Scheme is A-stable iff $|R(z)| < 1$ when $\operatorname{Re} z < 0$.

But $R(z) = -\frac{(\frac{1}{2} - a)}{a} z + \cdots$ for large $|z|$,

so divergent if $\frac{1}{2} \neq a$. Therefore NOT A-stable for $a \neq \frac{1}{2}$.

For $a = \frac{1}{2}$, $R(z) = \dfrac{1 + \frac{1}{2} z}{1 - \frac{1}{2} z}$

and $|R(z)| = \left| \dfrac{z - (-2)}{z - 2} \right| < 1$ for $\operatorname{Re} z < 0$,

so A-stable.