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17A Numerical Analysis

(a) Define *Householder reflections* and show that a real Householder reflection is a symmetric and orthogonal matrix.

(b) Let $H \in \mathbb{R}^{n\times n}$ be a Householder reflection. Determine the eigenvalues of H and their multiplicities.

(c) Show that for any $A \in \mathbb{R}^{n\times n}$ there exist Householder reflections $H_1, \ldots, H_n$ such that $H_n H_{n-1} \cdots H_1 A = R$, where R is upper triangular.

(d) Show that if A is symmetric there exists an orthogonal matrix $Q \in \mathbb{R}^{n\times n}$ such that $C = QAQ^T \in \mathbb{R}^{n\times n}$ is symmetric and tridiagonal (that is, only the diagonal, super and subdiagonal have non-zero entries), and C can be computed in finitely many operations $(+, -, \times, \div, \sqrt{\ })$.

(a) For any $w \in \mathbb{R}^n$ s.t. $\|w\| = 1$, we define

$$H = I - 2ww^T.$$

Then $H^T = H$ and

$$H^2 = (I - 2ww^T)(I - 2ww^T)$$
$$= I - 4ww^T + 4ww^Tww^T$$

$4ww^T$ since $w^Tw = 1$

$$= I.$$

Usually write $H = I - 2\dfrac{vv^T}{v^Tv}$ when $v \in \mathbb{R}^n \setminus \{0\}$.

(b) If $H = I - 2ww^T$, then

$$Hw = w - 2ww^Tw = w - 2w = -w$$

&, for $v \in w^\perp$,

$$Hv = v.$$

Hence the eigenvalues of H are $\begin{cases} -1 & \text{(unique)} \\ 1 & \text{(multiplicity } n-1\text{)}. \end{cases}$

(c) There is nothing to do when $A \in \mathbb{R}^{n\times n}$ and $n=1$.

For $A \in \mathbb{R}^{n\times n}$, $n>1$, we write

$$A = (a_1 \quad a_2 \quad \cdots \quad a_n), \qquad a_i \in \mathbb{R}^n,\ 1\le i\le n,$$

and we prove the following result.

LEMMA: If $\|\alpha\| = \|\beta\| \neq 0$, then let

$$w = \frac{\beta - \alpha}{\|\beta - \alpha\|}.$$

Then the reflection

$$H = I - 2ww^T$$

satisfies $H\alpha = \beta$.

Proof:

$$H\alpha = \alpha - \frac{2(\beta-\alpha)(\beta-\alpha)^T\alpha}{\|\beta-\alpha\|^2}$$

& $\|\beta-\alpha\|^2 = (\beta-\alpha)^T(\beta-\alpha) = 2(\|\alpha\|^2 - \alpha^T\beta)$,

so that

$$H\alpha = \frac{2\alpha(\|\alpha\|^2-\alpha^T\beta) - 2(\beta-\alpha)(\beta^T\alpha - \|\alpha\|^2)}{2(\|\alpha\|^2 - \alpha^T\beta)}$$

$$= \frac{\alpha\{\overbrace{2\|\alpha\|^2 - 2\alpha^T\beta + 2\beta^T\alpha - 2\|\alpha\|^2}^{=0}\} + 2\beta(\|\alpha\|^2 - \beta^T\alpha)}{2(\|\alpha\|^2 - \alpha^T\beta)}$$

$$= \beta. \qquad \square$$

We now choose $\alpha = a_1$, $\beta = \|a_1\| e_1$ if $a_1 \neq 0$

& let $H_1 = I - 2\dfrac{(\beta-\alpha)(\beta-\alpha)^T}{(\beta-\alpha)^T(\beta-\alpha)}$.

Then
$$A_1 \equiv H_1 A = \begin{pmatrix} \|a_1\| & x & \cdots & x \\ 0 & & & \\ \vdots & & \hat{A}_1 & \\ 0 & & & \end{pmatrix}.$$

We now choose $\hat{H}_2 \in \mathbb{R}^{(n-1)\times(n-1)}$ s.t.

$$\hat{H}_2 \hat{A}_1 = \begin{pmatrix} x & x & \cdots & x \\ 0 & & & \\ \vdots & & \hat{A}_2 & \\ 0 & & & \end{pmatrix}$$

& let

$$H_2 = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & & & \\ \vdots & & \hat{H}_2 & \\ 0 & & & \end{pmatrix}$$

so that

$$A_2 \equiv H_2 A_1 = \begin{pmatrix} \|a_1\| & x & \cdots & & x \\ 0 & x & x & \cdots & x \\ \vdots & 0 & & & \\ \vdots & \vdots & & \hat{A}_2 & \\ 0 & 0 & & & \end{pmatrix}.$$

We then proceed inductively to obtain

$$H_{n-1} \cdots H_2 H_1 A = R.$$

(d) Let

$$A = \begin{pmatrix} A_{11} & \alpha^T \\ \alpha & \hat{A} \end{pmatrix}.$$

If $\hat{H}_1 \in \mathbb{R}^{(n-1)\times(n-1)}$ satisfies $\hat{H}_1 \alpha = \begin{pmatrix} \|\alpha\| \\ 0 \\ \vdots \\ 0 \end{pmatrix}$,

then

$$H_1 = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & & & \\ \vdots & & \hat{H}_1 & \\ 0 & & & \end{pmatrix}$$

satisfies

$$H_1 A = \begin{pmatrix} 1 & \\ & \hat{H}_1 \end{pmatrix} \begin{pmatrix} A_{11} & \alpha^T \\ \alpha & \hat{A} \end{pmatrix}$$

$$= \begin{pmatrix} A_{11} & \alpha^T \\ \|\alpha\| & \\ 0 & \hat{H}_1 \hat{A} \\ \vdots & \\ 0 & \end{pmatrix}.$$

Hence

$$H_1 A H_1 = (H_1 A H)^T = H\,(H_1 A)^T$$

$$= \begin{pmatrix} 1 & \\ & \hat{H}_1 \end{pmatrix} \begin{pmatrix} A_{11} & \|\alpha\| \; 0 \; \cdots \; 0 \\ \alpha & (\hat{H}_1 \hat{A})^T \end{pmatrix}$$

$$= \begin{pmatrix} A_{11} & \|\alpha\| \; 0 \; \cdots \; 0 \\ \|\alpha\| & \\ 0 & \hat{H}_1 (\hat{H}_1 \hat{A})^T = \hat{H}_1 \hat{A} \hat{H}_1^T \\ \vdots & \\ 0 & \end{pmatrix}.$$

We then proceed inductively to obtain

$$H_{n-1} \cdots H_1 A H_1 \cdots H_{n-1} = T.$$