2025-2-17

17A Numerical Analysis

Consider the scalar autonomous ODE of the form

$$y' = f(y), \quad y(0) = y_0 \in \mathbb{R}, \tag{$*$}$$

where $y(t)$ exists and is unique for $t \in [0,T]$ and $T > 0$. Consider also the following two Runge-Kutta methods:

$$k_1 = f(y_n), \quad k_2 = f\big(y_n + \frac{h}{2}k_1 + \frac{h}{2}k_2\big), \quad y_{n+1} = y_n + \frac{h}{2}(k_1 + k_2), \tag{$\dagger$}$$

$$\text{same as } (\dagger) \text{ except } k_2 = f\big(y_n + \frac{h}{4}k_1 + \frac{3h}{4}k_2\big), \tag{$\ddagger$}$$

both producing a sequence $\{y_n\}_{n \leqslant N}$, where $N = \lfloor \frac{T}{h} \rfloor$ and $h > 0$ is the step-size.

(a) Do the above Runge-Kutta methods have the same order? If so, determine the order. If not, determine which method has the highest order.
[*Hint: Think about how both methods can be written in terms of a single parameter.*]

(b) For a numerical method approximating the solution of $(*)$, define the linear stability domain. What does it mean for such a numerical method to be A-stable?

(c) Are any of the Runge-Kutta methods $(\dagger)$ and $(\ddagger)$ A-stable? If so, determine the linear stability domain for the method(s).

ALMOST IDENTICAL TO 2022-2-17.