2025/3/17

17B Numerical Analysis

Consider $C[-1,1]$ equipped with the inner product $\langle f,g\rangle = \int_{-1}^{1} f(x)g(x)w(x)\,dx$, where $w(x)>0$ for $x\in(-1,1)$. Moreover, for $n\in\mathbb{N}$, let

$$A_n = \begin{bmatrix} \alpha_1 & \sqrt{\beta_2} & 0 & \cdots & 0 \\ \sqrt{\beta_2} & \alpha_2 & \sqrt{\beta_3} & \ddots & \vdots \\ 0 & \sqrt{\beta_3} & \alpha_3 & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & \sqrt{\beta_n} \\ 0 & \cdots & 0 & \sqrt{\beta_n} & \alpha_n \end{bmatrix},$$

where $\alpha_n\in\mathbb{R}$ and $\beta_n>0$.

(a) Let $\{p_n\}_{n=0}^{\infty}$ be a sequence of monic polynomials of degree n orthogonal with respect to the above inner product. Prove that for $n\geqslant 1$ each p_n has n distinct zeros in the interval $(-1,1)$.

(b) Let $P_0(x)=1$, $P_1(x)=x-\alpha_1$, and let P_n satisfy the following recurrence relation:

$$P_n(x) = (x-\alpha_n)P_{n-1}(x) - \beta_n P_{n-2}(x), \quad n\geqslant 2.$$

Prove that for $n>1$ we have $P_n(x)=\det(xI-A_n)$.

(c) Prove that if $p_0(x)=1$ and

$$\alpha_n = \frac{\langle p_n, xp_n\rangle}{\langle p_n,p_n\rangle}, \quad \beta_n = \frac{\langle p_n,p_n\rangle}{\langle p_{n-1},p_{n-1}\rangle}$$

then all the eigenvalues of A_n are distinct and reside in $(-1,1)$.

ERRORS: Should have $n \mapsto n-1$, $n-1 \mapsto n-2$. See below.

[*Hint: You may quote the three-term recurrence relation theorem from the class notes.*]

(a) $p_0(x) = 1$ and $p_1(x) = x - c_1$.

If $c_1 \notin (-1,1)$, i.e. if $|c_1| > 1$, then p_1 does not change sign in $[-1,1]$ and

$$\langle 1, p_1 \rangle = \int_{-1}^{1} p_1(x)\, w(x)\, dx$$

cannot vanish, since $|p_1(x)| \cdot w(x) > 0$ for $x \in [-1,1]$

Hence $c_1 \in (-1,1)$.

Now consider $p_n \in \mathbb{P}_n$. If p_n does not change sign in $(-1,1)$, then $\langle 1, p_1 \rangle \neq 0$, contradicting

$p_n \perp p_0$. Thus p_n changes sign at least once in $(-1, 1)$. Let

$$-1 < x_1 < x_2 < \cdots < x_M < 1$$

be the points at which p_n changes sign in $(-1, 1)$. We already know $M \geq 1$ and, if $M = n$, then we're done. Otherwise, if $M < n$, then $p_n(x)$ and

$$q(x) = (x - x_1)(x - x_2)\cdots(x - x_M)$$

always have the same sign, so

$$\langle p_n, q \rangle \neq 0.$$

But this contradicts $p_n \perp \mathbb{P}_{n-1}$ UNLESS $M = n$.

(b) $$A_n = \begin{pmatrix} \alpha_1 & \beta_2^{1/2} & & & \\ \beta_2^{1/2} & \alpha_2 & \beta_3^{1/2} & & \\ & \beta_3^{1/2} & \alpha_3 & \ddots & \\ & & \ddots & \ddots & \\ & & & \alpha_{n-1} & \beta_n^{1/2} \\ & & & \beta_n^{1/2} & \alpha_n \end{pmatrix} \in \mathbb{R}^{n \times n}.$$

Then $\chi_1(x) = x - \alpha_1$ and

$$\chi_2(x) \equiv \det(x . I - A_2) = \det \begin{pmatrix} x - \alpha_1 & -\beta_2^{1/2} \\ -\beta_2^{1/2} & x - \alpha_2 \end{pmatrix}.$$

$$= (x - \alpha_2)\chi_1(x) - \beta_2.$$

and so

$$\chi_2(x) = (x - \alpha_2)\chi_1(x) - \beta_2 \chi_0(x),$$

where $\chi_0(x) \equiv 1$.

More generally,

$$\chi_n(x) = \det(x\cdot I - A_n) = \begin{pmatrix} x-\alpha_1 & -\beta_2^{1/2} & & & & \\ -\beta_2^{1/2} & x-\alpha_2 & \ddots & & & \\ & \ddots & \ddots & \ddots & & \\ & & & x-\alpha_{n-2} & -\beta_{n-1}^{1/2} & \\ & & & -\beta_{n-1}^{1/2} & x-\alpha_{n-1} & -\beta_n^{1/2} \\ & & & & -\beta_n^{1/2} & x-\alpha_n \end{pmatrix}$$

So, expanding det along final row gives

$$\chi_n(x) = (x-\alpha_n)\,\chi_{n-1}(x) - (-\beta_n^{1/2})\,\det \begin{pmatrix} x-\alpha_1 & -\beta_2^{1/2} & & & \\ -\beta_2^{1/2} & \ddots & \ddots & & \\ & \ddots & & -\beta_{n-2}^{1/2} & \\ & & -\beta_{n-2}^{1/2} & x-\alpha_{n-2} & \\ & & & -\beta_{n-1}^{1/2} & -\beta_n^{1/2} \end{pmatrix}$$

$$\chi_n(x) = (x-\alpha_n)\,\chi_{n-1}(x) - \beta_n\,\chi_{n-2}(x).$$

Thus $\chi_n \equiv P_n$, since they satisfy the same recurrence and $\chi_0 = P_0$, $\chi_1 = P_1$.

(c) The 3-term recurrence for (P_n) is

$$P_{n+1}(x) = (x - A_n)\,P_n(x) - B_n\,P_{n-1}(x), \quad n \geq 1,$$

where $A_n = \dfrac{\langle xP_n, P_n\rangle}{\langle P_n, P_n\rangle}$ and $B_n = \dfrac{\langle P_n, P_n\rangle}{\langle P_{n-1}, P_{n-1}\rangle}$.

OR $(k = n+1)$

$$P_k(x) = (x - \alpha_k)\,P_{k-1}(x) - \beta_k\,P_{k-2}(x), \quad k \geq 2,$$

where $\alpha_k = \dfrac{\langle xP_{k-1}, P_{k-1}\rangle}{\langle P_{k-1}, P_{k-1}\rangle}$, $\beta_k = \dfrac{\langle P_{k-1}, P_{k-1}\rangle}{\langle P_{k-2}, P_{k-2}\rangle}$.

HENCE ERROR IN QUESTION.

If $\alpha_n = \dfrac{\langle x p_{n-1}, p_{n-1}\rangle}{\langle p_{n-1}, p_{n-1}\rangle}$ and $\beta_n = \dfrac{\langle p_{n-1}, p_{n-1}\rangle}{\langle p_{n-2}, p_{n-2}\rangle}$

then we have the 3-term recurrence relation for the orthogonal polynomials w.r.t. the inner product

$$\langle F, G\rangle = \int_{-1}^{1} F(x)\, G(x)\, w(x)\, dx.$$

Thus

$$p_n(x) = \det(x.I - A_n)$$

has n distinct roots in $(-1,1)$, by (a), and these are exactly the eigenvalues of A_n.

Note that $P_1(x) = \det(xI - A_1) = x - \alpha_1$

where $\alpha_1 = \dfrac{\langle x p_0, p_0\rangle}{\langle p_0, p_0\rangle} = \dfrac{\langle x, 1\rangle}{\langle 1, 1\rangle}$

and $\langle 1, P_1\rangle = 0$, so it really is the 3-term recurrence.

[Note that the given formula $\alpha_1 = \dfrac{\langle x p_1, p_1\rangle}{\langle p_1, p_1\rangle}$, which is incorrect.]

OPTIONAL EXTRA (NOT needed in EXAM): 3-term recurrence:

$$\phi_0(x) = 1, \quad \phi_1(x) = x - c_1, \quad \phi_n \in \mathbb{P}_n \text{ monic.}$$

Then

$$\phi_{n+1}(x) - x\phi_n(x) \in \mathbb{P}_n$$

and

$$\phi_{n+1} - x\phi_n = \sum_{k=0}^{n} \mu_k \phi_k$$

where

$$\mu_k = \frac{\langle \phi_{n+1} - x\phi_n, \phi_k \rangle}{\langle \phi_k, \phi_k \rangle}, \quad 0 \le k \le n-1.$$

Now $\langle \phi_{n+1}, \phi_k \rangle = 0$ for $k < n$, so

$$\mu_k = -\frac{\langle \phi_n, x\phi_k \rangle}{\langle \phi_k, \phi_k \rangle}.$$

But $\langle \phi_n, x\phi_k \rangle = 0$ if $k+1 < n$.

Thus only μ_{n-1} and μ_n are nonzero.

We have

$$\mu_{n-1} = -\frac{\langle \phi_n, x\phi_{n-1} \rangle}{\langle \phi_{n-1}, \phi_{n-1} \rangle}$$

and

$$\mu_n = -\frac{\langle \phi_n, x\phi_n \rangle}{\langle \phi_n, \phi_n \rangle}.$$

Now,

$$\langle \phi_n, x\phi_{n-1} \rangle = \langle \phi_n, \underbrace{x\phi_{n-1} - \phi_n}_{\in \mathbb{P}_{n-1}} \rangle + \langle \phi_n, \phi_n \rangle$$

$$= \langle \phi_n, \phi_n \rangle,$$

so

$$\mu_{n-1} = -\frac{\langle \phi_n, \phi_n \rangle}{\langle \phi_{n-1}, \phi_{n-1} \rangle}.$$

Thus, given $\phi_0(x) = 1$ and $\phi_1(x) = x - c_1$, where $c_1 = \langle 1, x\rangle / \langle x, x\rangle$, we have

$$\phi_{n+1}(x) - x\phi_n(x) = -\frac{\langle \phi_n, x\phi_n\rangle}{\langle \phi_n, \phi_n\rangle}\phi_n(x) - \frac{\langle \phi_n, \phi_n\rangle}{\langle \phi_{n-1}, \phi_{n-1}\rangle}\phi_{n-1}(x)$$

i.e.

$$\phi_{n+1}(x) = \left(x - \frac{\langle \phi_n, x\phi_n\rangle}{\langle \phi_n, \phi_n\rangle}\right)\phi_n(x) - \frac{\langle \phi_n, \phi_n\rangle}{\langle \phi_{n-1}, \phi_{n-1}\rangle}\phi_{n-1}(x)$$

for $n \geq 1$.