2025-4-6

Paper 4, Section I

6A Numerical Analysis

Consider the quadrature formula

$$\int_0^1 f(x)x\,dx \approx \sum_{i=0}^{1} a_i f(x_i), \quad x_i \in [0,1], \qquad f \in C[0,1], \qquad (*)$$

which is exact for polynomials of degree 1.

(a) For $i = 0, 1$, find expressions for the weights a_i in terms of the nodes x_0, x_1.

(b) Define what it means for $(*)$ to be a *Gaussian quadrature*, and determine the numerical values of the nodes x_0, x_1 in that case.

(a) $f = 1 \Rightarrow \int_0^1 x\,dx = \frac{1}{2} = a_0 + a_1.$

$f = x \Rightarrow \int_0^1 x^2\,dx = \frac{1}{3} = a_0 x_0 + a_1 x_1.$

So

$$\begin{pmatrix} 1 & 1 \\ x_0 & x_1 \end{pmatrix}\begin{pmatrix} a_0 \\ a_1 \end{pmatrix} = \begin{pmatrix} 1/2 \\ 1/3 \end{pmatrix}$$

ASSUME $x_0 < x_1$

$$\rightarrow \begin{pmatrix} a_0 \\ a_1 \end{pmatrix} = \frac{1}{x_1 - x_0}\begin{pmatrix} x_1 & -1 \\ -x_0 & 1 \end{pmatrix}\begin{pmatrix} 1/2 \\ 1/3 \end{pmatrix}$$

or $a_0 = \dfrac{\frac{1}{2}x_1 - \frac{1}{3}}{x_1 - x_0}$, $a_1 = \dfrac{\frac{1}{3} - \frac{1}{2}x_0}{x_1 - x_0}.$

CHECK: $a_0 + a_1 = \dfrac{\frac{1}{2}x_1 - \frac{1}{3} + \frac{1}{3} - \frac{1}{2}x_0}{x_1 - x_0} = \frac{1}{2}$ ✓

$$a_0 x_0 + a_1 x_1 = \frac{(\frac{1}{2}x_1 - \frac{1}{3})x_0 + (\frac{1}{3} - \frac{1}{2}x_0)x_1}{x_1 - x_0}$$

$= \frac{1}{3}.$ ✓

(b) For Gaussian quadrature, we choose x_0, x_1 to be the roots of the quadratic orthogonal poly

$$\phi_2(x) = (x - x_0)(x - x_1)$$

w.r.t. the inner product

$$\langle f, g \rangle = \int_0^1 x\, f(x)\, g(x)\, dx.$$

Now $\phi_0(x) = 1$ & $\phi_1(x) = x - \alpha$, where

$$0 = \langle \phi_0, \phi_1 \rangle = \int_0^1 x(x - \alpha)\, dx$$

$$= \frac{1}{3} - \frac{1}{2}\alpha$$

$\Rightarrow \quad \alpha = \frac{2}{3}$, i.e. $\phi_1(x) = x - \frac{2}{3}$.

Then $\phi_2(x) = x^2 + \alpha_1 x + \alpha_0$

where

$$0 = \langle 1, \phi_2 \rangle = \langle 1, x^2 \rangle + \alpha_1 \langle 1, x \rangle + \alpha_0 \langle 1, 1 \rangle$$

$$0 = \langle x, \phi_2 \rangle = \langle x, x^2 \rangle + \alpha_1 \langle x, x \rangle + \alpha_0 \langle x, 1 \rangle.$$

I.e.

$$\begin{pmatrix} \langle 1, 1 \rangle & \langle 1, x \rangle \\ \langle 1, x \rangle & \langle x, x \rangle \end{pmatrix} \begin{pmatrix} \alpha_0 \\ \alpha_1 \end{pmatrix} = - \begin{pmatrix} \langle 1, x^2 \rangle \\ \langle x, x^2 \rangle \end{pmatrix}$$

OR

$$\begin{pmatrix} \frac{1}{2} & \frac{1}{3} \\ \frac{1}{3} & \frac{1}{4} \end{pmatrix} \begin{pmatrix} \alpha_0 \\ \alpha_1 \end{pmatrix} = - \begin{pmatrix} 1/4 \\ 1/5 \end{pmatrix}$$

Then
$$\begin{pmatrix}\alpha_0\\ \alpha_1\end{pmatrix} = \frac{-1}{\underbrace{\frac{1}{2\cdot 4} - \frac{1}{3^2}}_{\frac{9-8}{72}}} \underbrace{\begin{pmatrix}\frac{1}{4} & -\frac{1}{3}\\ -\frac{1}{3} & \frac{1}{2}\end{pmatrix}\begin{pmatrix}1/4\\ 1/5\end{pmatrix}}_{\begin{pmatrix}\frac{1}{4^2} - \frac{1}{3.5}\\ -\frac{1}{3.4} + \frac{1}{2.5}\end{pmatrix}}$$

$$= -72 \begin{pmatrix}\frac{3.5 - 4^2}{3.4^2.5}\\ \frac{-2.5 + 3.4}{2.3.4.5}\end{pmatrix}$$

$$= -72 \begin{pmatrix}\frac{-1}{3.4^2.5}\\ \frac{1}{3.4.5}\end{pmatrix}$$

$$\frac{72}{3.4^2.5} = \frac{6}{20} = \frac{3}{10}$$

$$= \frac{72}{3.4^2.5}\begin{pmatrix}1\\ -4\end{pmatrix}$$

$$\begin{pmatrix}\alpha_0\\ \alpha_1\end{pmatrix} = \frac{3}{10}\begin{pmatrix}1\\ -4\end{pmatrix}.$$

Thus $\phi_2(x) = x^2 - \frac{6}{5}x + \frac{3}{10}$.

Hence
$$0 = \phi_2(x) = \left(x - \frac{6}{10}\right)^2 + \underbrace{\frac{3}{10} - \frac{6^2}{100}}_{\frac{30-36}{100}}$$
$$= \frac{-6}{100}$$

Thus $x = \frac{6}{10} \pm \frac{\sqrt{6}}{10} = \frac{1}{10}(6 \pm \sqrt{6})$.