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5A Numerical Analysis

Consider the ODE of the form

$$y'(t) = f(t, y(t)), \quad y(0) = y_0 \in \mathbb{R}, \qquad (*)$$

where $y(t)$ exists and is unique for $t \in [0, T]$ and $T > 0$.

(a) State the Dahlquist equivalence theorem regarding convergence of a multistep method.

(b) Consider the following multistep method for $(*)$ with a parameter $\alpha \in \mathbb{R}$:

SHOULD BE $+$

$$y_{n+3} + (2\alpha - 3)(y_{n+2} - y_{n+1}) - y_n = h\alpha\big(f(t_{n+2}, y_{n+2}) - f(t_{n+1}, y_{n+1})\big),$$

producing a sequence $\{y_n\}_{n\leqslant N}$, where $N = \lfloor \frac{T}{h} \rfloor$ and $h > 0$ is the step-size. It is given that the method is of at least order 2 for any α and also of order 3 for $\alpha = 6$. Determine all values of α for which the method is convergent, and find the order of convergence.

(a) Dahlquist Equivalence Theorem:

The Multistep method

$$\sum_{l=0}^{s} \rho_l y_{n+l} = h \sum_{l=0}^{s} \sigma_l f(t_{n+l}, y_{n+l})$$

is convergent iff it is of order $p \geq 1$ AND

$$\rho(z) = \sum_{l=0}^{s} \rho_l y_{n+l}$$

satisfies the ROOT CONDITION: every root of ρ is in $|z| \leq 1$ & any roots on $|z| = 1$ are simple.

(b) We only need to check the root condition, where

$$\rho(z) = z^3 + (2\alpha - 3)(z^2 - z) - 1.$$

Then $\rho(1) = 0$ so we divide ρ by $z - 1$:

$$\begin{array}{r|l} & z^2 + (2\alpha-2)z + 1 \\ \hline z-1 & z^3 + (2\alpha-3)z^2 - (2\alpha-3)z - 1 \\ & z^3 - z^2 \end{array}$$

$$\begin{array}{r} (2\alpha-2)z^2 - (2\alpha-3)z \\ (2\alpha-2)z^2 - (2\alpha-2)z \\ \hline z-1 \end{array}$$

Thus $\rho(z) = (z-1)\,(\underbrace{z^2 + 2(\alpha-1)z + 1}_{q(z)})$

& the roots z_1, z_2 of q are given by

$$z = 1-\alpha \pm \sqrt{(\alpha-1)^2 - 1}.$$

If the roots are real, then $z_1 z_2 = 1$, so

either at least one has modulus > 1

or $z_1 = z_2 = 1$ } i.e. double roots.

or $z_1 = z_2 = -1$

Hence ρ cannot satisfy the root condition for real roots & we need a nonreal complex conjugate pair, so

$$(\alpha-1)^2 - 1 < 0$$

i.e.

$$(\alpha-1)^2 < 1$$

or

$$-1 < \alpha - 1 < 1$$

i.e.

$$0 < \alpha < 2$$

If $0 < \alpha < 2$, then $z_1 = e^{i\theta}$, $z_2 = e^{-i\theta}$, $\theta \neq 0$ or π, so CONVERGENT, since question states that method is order 2 & we apply Dahlquist.

BUT the given multistep method is NOT of order 2: just try $y = x^2$. I have included the order below for interest.

ORDER: We substitute $y = 1, x, x^2, x^3$ in

$$y(3) + (2\alpha - 3)(y(2) - y(1)) - y(0) = \alpha(y'(2) + y'(1));$$

$y(x)$	LHS	RHS
1	0	0 ✓
x	$3 + 2\alpha - 3 = 2\alpha$	2α ✓
x^2	$3^2 + (2\alpha - 3)(2^2 - 1) = 6\alpha$	$2\alpha(2+1) = 6\alpha$ ✓
x^3	$3^3 + (2\alpha - 3)(2^3 - 1) = 14\alpha + 6$	$3\alpha(2^2 + 1^2) = 15\alpha$

So alway order 2 & order 3 iff $\alpha = 6$.

BUT method only satisfies root condition for $0 < \alpha < 2$, so convergent iff $0 < \alpha < 2$ & order 2.