

Mathematical Tripos Part IB: Lent Term 2024

Numerical Analysis – Lecture 1

1 Polynomial interpolation

1.1 Lagrange and Newton polynomials

Let $f : [a, b] \rightarrow \mathbb{R}$ be a real-valued continuous function defined on some interval $[a, b]$ and let $(x_i)_{i=0}^n$ be $n + 1$ distinct points in $[a, b]$. We wish to construct a polynomial p of degree n (notation $p \in \mathcal{P}_n$) which interpolates f at these points, i.e., satisfies

$$p(x_i) = f(x_i), \quad i = \overline{0, n}, \quad p \in \mathcal{P}_n.$$

Theorem 1.1 (Existence and uniqueness) *Given $f \in C[a, b]$ and $n + 1$ distinct points $(x_i)_{i=0}^n \in [a, b]$, there is exactly one polynomial $p \in \mathcal{P}_n$ such that $p(x_i) = f(x_i)$ for all i .*

Proof. 1) There is *at least one* polynomial interpolant $p \in \mathcal{P}_n$, the one in the *Lagrange form*,

$$p(x) = \sum_{i=0}^n f(x_i) \ell_i(x) \quad \text{with} \quad \ell_i(x) := \prod_{\substack{j=0 \\ j \neq i}}^n \frac{x - x_j}{x_i - x_j}, \quad i = \overline{0, n}, \quad (1.1)$$

where ℓ_i s are called the *fundamental* Lagrange polynomials. Each ℓ_i is the product of n linear factors, hence $\ell_i \in \mathcal{P}_n$. It also equals 1 at x_i and vanishes at $x_j \neq x_i$, i.e., $\ell_i(x_j) = \delta_{ij}$. Therefore $p \in \mathcal{P}_n$ and

$$p(x_j) = \sum_{i=0}^n f(x_i) \ell_i(x_j) = f(x_j).$$

2) There is *at most one* polynomial interpolant $p \in \mathcal{P}_n$ to f on $(x_i)_{i=0}^n$. For if there are two, $p, q \in \mathcal{P}_n$, then the polynomial $r := p - q$ is of degree n and vanishes at $n + 1$ points, whence $r \equiv 0$. $\square$

Remark 1.2 Let us introduce the so-called *nodal* polynomial

$$\omega(x) := \prod_{i=0}^n (x - x_i).$$

Then, in the expression (1.1) for ℓ_i , the numerator is simply $\omega(x)/(x - x_i)$ while the denominator is equal to $\omega'(x_i)$. With that we arrive to a compact Lagrange form

$$p(x) = \sum_{i=0}^n f(x_i) \ell_i(x) = \sum_{i=0}^n \frac{f(x_i)}{\omega'(x_i)} \frac{\omega(x)}{x - x_i} \quad (1.2)$$

The Lagrange forms (1.1)-(1.2) for the interpolating polynomials are easy to manipulate theoretically, but they are unsuitable for numerical evaluation (unstable). A better alternative is the *Newton form* which has an *adaptive* nature.

Method 1.3 (The Newton form) For $k = 0, 1, \dots, n$, let $p_k \in \mathcal{P}_k$ be the polynomial interpolant to f on $x_0, \dots, x_k$. Then two subsequent p_{k-1} and p_k interpolate the same values $f(x_i)$ for $0 \leq i \leq k-1$, hence their difference is a polynomial of degree k that vanishes at k points $x_0, \dots, x_{k-1}$. Thus

$$p_k(x) - p_{k-1}(x) = A_k \prod_{i=0}^{k-1} (x - x_i), \quad (1.3)$$

with some constant A_k which is seen to be equal to the *leading* coefficient of p_k . It follows that $p := p_n$ can be built step by step as one constructs the sequence $(p_0, p_1, \dots)$, with p_k obtained from p_{k-1} by adding the term from the right-hand side of (1.3), so that finally

$$p(x) := p_n(x) = p_0(x) + \sum_{k=1}^n [p_k(x) - p_{k-1}(x)] = \sum_{k=0}^n A_k \prod_{i=0}^{k-1} (x - x_i).$$

Definition 1.4 (Divided difference) Given $f \in C[a, b]$ and $k+1$ distinct points $(x_i)_{i=0}^k \in [a, b]$, the *divided difference* $f[x_0, \dots, x_k]$ of order k is, by definition, the leading coefficient of the polynomial $p_k \in \mathcal{P}_k$ that interpolates f at these points. By definition, it is a symmetric function of the variables $[x_0, \dots, x_k]$, and if $f(x) = x^m$, $0 \leq m \leq k$, then $f[x_0, \dots, x_k] = \delta_{km}$.

With this definition we arrive at the Newton formula for the interpolating polynomial.

Theorem 1.5 (Newton formula) Given $n+1$ distinct points $(x_i)_{i=0}^n$, let $p_n \in \mathcal{P}_n$ be the polynomial that interpolates f at these points. Then it may be written in the Newton form

$$\begin{aligned} p_n(x) = & f[x_0] + f[x_0, x_1](x - x_0) + f[x_0, x_1, x_2](x - x_0)(x - x_1) + \dots \\ & \dots + f[x_0, x_1, \dots, x_n](x - x_0)(x - x_1) \dots (x - x_{n-1}), \end{aligned}$$

or, more compactly,

$$p_n(x) = \sum_{k=0}^n f[x_0, \dots, x_k] \prod_{i=0}^{k-1} (x - x_i) \quad (1.4)$$

To make this formula of any use, we need an expression for $f[x_0, \dots, x_k]$. One such can be derived from the Lagrange formula (1.2) by identifying the leading coefficient of p . This turns out to be

$$f[x_0, \dots, x_n] = \sum_{i=0}^n \frac{f(x_i)}{\omega'(x_i)}, \quad \omega(x) := \prod_{i=0}^n (x - x_i).$$

However, this expression has the same computational disadvantages as the Lagrange form itself (instability). A useful way to calculate divided difference is again an adaptive (or recurrence) approach.

Theorem 1.6 (Recurrence relation) For distinct $x_0, x_1, \dots, x_k$ (with $k \geq 1$), we have

$$f[x_0, \dots, x_k] = \frac{f[x_1, \dots, x_k] - f[x_0, \dots, x_{k-1}]}{x_k - x_0} \quad (1.5)$$

Proof. Let $q_0, q_1 \in \mathcal{P}_{k-1}$ be the polynomials such that q_0 interpolates f on $(x_0, x_1, \dots, x_{k-1})$ and q_1 interpolates f on $(x_1, \dots, x_{k-1}, x_k)$ and consider the polynomial

$$p(x) := \frac{x - x_0}{x_k - x_0} q_1(x) + \frac{x_k - x}{x_k - x_0} q_0(x), \quad p \in \mathcal{P}_k.$$

One readily sees that $p(x_i) = f(x_i)$ for all i , hence, p is the k -th degree interpolating polynomial for f . Moreover, the leading coefficient of p is equal to the difference of those of q_1 and q_0 divided by $x_k - x_0$, and that is exactly what the recurrence (1.5) says. $\square$

Method 1.7 The recursive formula (1.5) allows for fast evaluation of the *divided difference table*

x_i	$f[*] = f(*)$	$f[*, *]$	$f[*, *, *]$	$f[*, *, *, *]$
x_0	$\rightarrow f[x_0]$	$\searrow f[x_0, x_1]$		
x_1	$\rightarrow f[x_1]$	$\searrow f[x_1, x_2]$	$\searrow f[x_0, x_1, x_2]$	
x_2	$\rightarrow f[x_2]$	$\searrow f[x_2, x_3]$	$\searrow f[x_1, x_2, x_3]$	$\searrow f[x_0, x_1, x_2, x_3]$
x_3	$\rightarrow f[x_3]$			

This can be done in $\mathcal{O}(n^2)$ operations and the outcome is the numbers $\{f[x_0, \dots, x_k]\}_{k=0}^n$ at the head of the columns which can be used in the Newton form (1.4).

Method 1.8 (Horner scheme) Finally, evaluation of p at a given point x using Newton formula (provided that divided differences $A_k := f[x_0, \dots, x_k]$ are known) requires just n multiplications, as long as we do it by the *Horner scheme*

$$p_n(x) = \left\{ \cdots \left\{ A_n \times (x - x_{n-1}) + A_{n-1} \right\} \times (x - x_{n-2}) + A_{n-2} \right\} \times (x - x_{n-3}) + \cdots + A_1 \times (x - x_0) + A_0.$$

1.2 Examples

Example 1.9 Given the data

x_i	0	1	2	3
$f(x_i)$	-3	-3	-1	9

find the interpolating polynomial $p \in \mathcal{P}_3$ in both Lagrange and Newton forms.

1a) Fundamental Lagrange polynomials

$$\begin{aligned} \ell_0(x) &= \frac{(x-1)(x-2)(x-3)}{-6} = -\frac{1}{6}(x^3 - 6x^2 + 11x - 6), \\ \ell_1(x) &= \frac{x(x-2)(x-3)}{2} = \frac{1}{2}(x^3 - 5x^2 + 6x), \\ \ell_2(x) &= \frac{x(x-1)(x-3)}{-2} = -\frac{1}{2}(x^3 - 4x^2 + 3x), \\ \ell_3(x) &= \frac{x(x-1)(x-2)}{6} = \frac{1}{6}(x^3 - 3x^2 + 2x). \end{aligned}$$

1b) Lagrange form:

$$\begin{aligned} p(x) &= (-3) \cdot \ell_0(x) + (-3) \cdot \ell_1(x) + (-1) \cdot \ell_2(x) + 9 \cdot \ell_3(x) \\ &= \left(\frac{1}{2} - \frac{3}{2} + \frac{1}{2} + \frac{3}{2} \right) x^3 + \left(-3 + \frac{15}{2} - 2 - \frac{9}{2} \right) x^2 + \left(\frac{11}{2} - 9 + \frac{3}{2} + 3 \right) x - 3 \\ &= x^3 - 2x^2 + x - 3. \end{aligned}$$

2a) Divided differences:

$$\begin{array}{l} \hat{0} \quad -3 \quad \searrow \quad \frac{(-3)-(-3)}{1-0} = 0^* \quad \searrow \quad \frac{2-0}{2-0} = 1^* \quad \searrow \quad \frac{4-1}{3-0} = 1^* \\ \hat{1} \quad -3 \quad \nearrow \quad \frac{(-1)-(-3)}{2-1} = 2 \quad \nearrow \quad \frac{10-2}{3-1} = 4 \quad \nearrow \\ \hat{2} \quad -1 \quad \searrow \quad \frac{9-(-1)}{3-2} = 10 \quad \nearrow \\ \hat{3} \quad 9 \end{array}$$

2b) Newton form:

$$p(x) = -\hat{3} + 0^* \cdot (x - \hat{0}) + 1^* \cdot (x - \hat{0})(x - \hat{1}) + 1^* \cdot (x - \hat{0})(x - \hat{1})(x - \hat{2}).$$

2c) Horner scheme

$$p(x) = \left\{ \left[1^* \cdot (x - \hat{2}) + 1^* \right] \cdot (x - \hat{1}) + 0^* \right\} \cdot (x - \hat{0}) - \hat{3}.$$

1.3 Some further formulas

Theorem 1.10 Let $p_n \in \mathcal{P}_n$ interpolate $f \in C[a, b]$ at $n+1$ distinct points $x_0, \dots, x_n$. Then for any $x \notin (x_i)$

$$f(x) - p_n(x) = f[x_0, \dots, x_n, x] \omega(x) \quad (1.6)$$

Proof. Given $x_0, \dots, x_n$, let $\bar{x} := x_{n+1}$ be any other point. Then, by (1.3), the corresponding polynomials p_n and p_{n+1} are related by

$$p_{n+1}(x) = p_n(x) + f[x_0, \dots, x_n, \bar{x}] \omega(x), \quad \omega(x) := \prod_{i=0}^n (x - x_i).$$

In particular, putting $x = \bar{x}$, and noticing that $p_{n+1}(\bar{x}) = f(\bar{x})$, we obtain

$$f(\bar{x}) = p_n(\bar{x}) + f[x_0, \dots, x_n, \bar{x}] \omega(\bar{x}),$$

the latter equality being the same as (1.6). $\square$

This theorem shows the error to be "like the next term" in the Newton form. However, we cannot evaluate the right-hand side of (1.6) without knowing the number $f(x)$ (which we are trying to approximate). But as we now show we can relate it to the $(n+1)$ -st derivative of f (for which we may have some upper bound). For this we need a version of the Rolle's theorem:

Lemma 1.11 If $g \in C^k[a, b]$ is zero at $k + \ell$ distinct points, then $g^{(k)}$ has at least ℓ distinct zeros in $[a, b]$.

Proof. By Rolle's theorem, if $\phi \in C^1$ is zero at two points, then ϕ' is zero at an intermediate point. So, we deduce that g' vanishes at least at $(k-1) + \ell$ distinct points. Next, applying Rolle to g' , we conclude that g'' vanishes at $(k-2) + \ell$ points, and so on. $\square$

Theorem 1.12 Let $[\bar{a}, \bar{b}]$ be the smallest interval that contains $x_0, \dots, x_k$ and let $f \in C^k[\bar{a}, \bar{b}]$. Then there exists $\xi \in [\bar{a}, \bar{b}]$ such that

$$f[x_0, \dots, x_k] = \frac{1}{k!} f^{(k)}(\xi) \quad (1.7)$$

Proof. Let $p \in \mathcal{P}_k$ be the interpolating polynomial to f on (x_i) . The error function $f - p$ has at least $k+1$ zeros in $[\bar{a}, \bar{b}]$ so, by Rolle's theorem, $f^{(k)} - p^{(k)}$ must vanish at some $\xi \in [\bar{a}, \bar{b}]$, i.e.,

$$p^{(k)}(\xi) = f^{(k)}(\xi).$$

On the other hand, if $p(x) = a_k x^k + (\text{lower order terms})$, then (for any ξ)

$$p^{(k)}(\xi) = k! a_k =: k! f[x_0, \dots, x_n]. \quad \square$$