

Mathematical Tripos Part IB: Lent Term 2024

Numerical Analysis – Lecture 2

2 Error bounds for polynomial interpolation

Here we study the *interpolation error*

$$e_n(x) = f(x) - p_n(x), \quad p_n \in \mathcal{P}_n,$$

for the class of differentiable functions f that possess $n + 1$ continuous derivatives on the interval $[a, b]$; we denote this class by $C^{n+1}[a, b]$.

Theorem 2.1 *Let $f \in C^{n+1}[a, b]$, and let $p_n \in \mathcal{P}_n$ interpolate f at $n + 1$ points $(x_i)_{i=0}^n \in [a, b]$, and let $\omega(x) := \prod_{i=0}^n (x - x_i)$. Then for every $x \in [a, b]$ there exists $\xi \in [a, b]$ such that*

$$f(x) - p_n(x) = \frac{1}{(n+1)!} \omega(x) f^{(n+1)}(\xi) \quad (2.1)$$

Proof. If x coincides with any x_i from the interpolating set, then both sides of (2.1) vanish, hence the formula trivially holds. So, we let x be any other fixed point, and consider the function

$$\phi(t) := [f(t) - p(t)] - c_x \omega(t),$$

with some constant c_x . For any c_x , $\phi(t) = 0$ at $n + 1$ points $t = x_i$, and we choose particular c_x so that $\phi(t) = 0$ at $t = x$ as well, i.e.,

$$c_x := \frac{f(x) - p(x)}{\omega(x)}.$$

Then ϕ has $n + 2$ distinct zeros hence, by Rolle's theorem, $\phi^{(n+1)}(\xi) = 0$ for some $\xi \in [a, b]$. So,

$$0 = \phi^{(n+1)}(\xi) = [f^{(n+1)}(\xi) - p^{(n+1)}(\xi)] - c_x \omega^{(n+1)}(\xi) = f^{(n+1)}(\xi) - c_x (n+1)!,$$

whence

$$c_x := \frac{f(x) - p(x)}{\omega(x)} = \frac{1}{(n+1)!} f^{(n+1)}(\xi),$$

and that is the same as (2.1). □

The equality (2.1) with the value $f^{(n+1)}(\xi)$ for some ξ is of hardly any use. Usually one has a bound for $f^{(n+1)}$ in terms of some *norm*, e.g., the L_∞ -norm (the *max-norm*)

$$\|g\|_\infty := \|g\|_{L_\infty[a,b]} := \max_{t \in [a,b]} |g(t)|.$$

Then estimate (2.1) takes the form

$$|f(x) - p_n(x)| \leq \frac{1}{(n+1)!} |\omega(x)| \|f^{(n+1)}\|_\infty \quad (2.2)$$

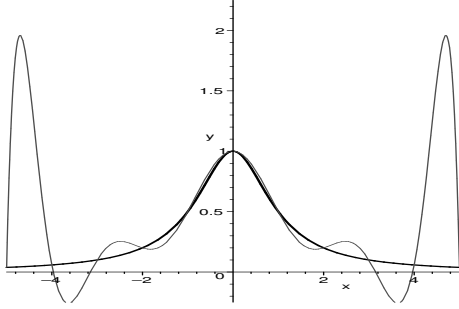
If we want to find the maximal error over the interval, then maximizing first the right- and then the left-hand side over $x \in [a, b]$ we get yet one more error bound for polynomial interpolation

$$\|f - p_\Delta\|_\infty \leq \frac{1}{(n+1)!} \|\omega_\Delta\|_\infty \|f^{(n+1)}\|_\infty \quad (2.3)$$

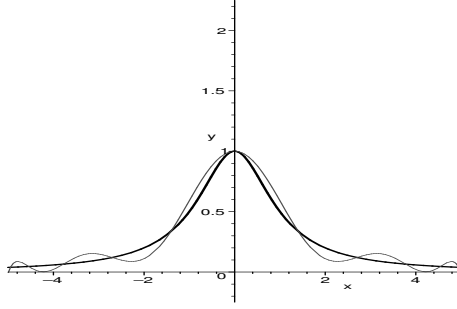
Here we put the lower index in ω_Δ in order to emphasize dependence of $\omega(x) := \prod_{i=0}^n (x - x_i)$ on the sequence of interpolating points $\Delta := (x_i)_{i=0}^n$. The right choice of Δ makes a big difference!

Example 2.2

$$f(x) = \frac{1}{1+x^2}, \quad x \in [-5, 5].$$



a) Interpolation at uniform knots $\{-5+k\}_{k=0}^{10}$



b) Interpolation at Chebyshev knots $\{-5 \cos \frac{2k+1}{22} \pi\}_{k=0}^{10}$

Definition 2.3 The Chebyshev polynomial of degree n on $[-1, 1]$ is defined by

$$T_n(x) = \cos n\theta, \quad x = \cos \theta, \quad \theta \in [0, \pi].$$

(Or just $T_n(x) = \cos n \arccos x$, with $x \in [-1, 1]$.) One sees at once that, on $[-1, 1]$,

1) T_n takes its maximal absolute value 1 with alternating signs $n+1$ times:

$$\|T_n\|_\infty = 1, \quad T_n(t_k) = (-1)^k, \quad t_k = \cos \frac{\pi k}{n}, \quad k = \overline{0, n};$$

2) T_n has n distinct zeros: $T_n(x_i^*) = 0$, $x_i^* = \cos \frac{2i+1}{2n} \pi$, $i = \overline{0, n-1}$.

Lemma 2.4 The Chebyshev polynomials T_n satisfy the three-term recurrence relation

$$T_0(x) \equiv 1, \quad T_1(x) = x, \tag{2.4}$$

$$T_{n+1}(x) = 2xT_n(x) - T_{n-1}(x), \quad n \geq 1. \tag{2.5}$$

In particular, T_n is indeed an algebraic polynomial of degree n with the leading coefficient 2^{n-1} .

Proof. Expressions (2.4) are straightforward, the recurrence (2.5) follows via substitution $x = \cos \theta$ into identity $\cos(n+1)\theta + \cos(n-1)\theta = 2 \cos \theta \cos n\theta$. $\square$

Theorem 2.5 On the interval $[-1, 1]$, among all polynomials of degree n with the leading coefficient equal to one, the Chebyshev polynomial γT_n has the smallest max-norm, i.e.,

$$\inf_{(a_i)} \|x^n + a_{n-1}x^{n-1} + \dots + a_0\|_\infty = \|\gamma T_n\|_\infty = \gamma, \quad \gamma := 1/2^{n-1}. \tag{2.6}$$

Proof. Suppose there is a polynomial $q_n(x) = x^n + a_{n-1}x^{n-1} + \dots + a_0$ such that $\|q\|_\infty < \gamma$, and set

$$r := \gamma T_n - q_n.$$

1) The leading coefficients of both q_n and γT_n are equal 1, thus r is of degree at most $n-1$.

2) Further, at $n+1$ points $t_k := \cos \frac{\pi k}{n}$, the Chebyshev polynomial γT_n takes the values $\pm \gamma$ alternatively, while by assumption $|q_n(t_k)| < \gamma$, hence $r = \gamma T_n - q_n$ alternates in sign at these $n+1$ points, therefore it has a zero in each of n intervals (t_k, t_{k+1}) , i.e. at least n zeros in the interval $[-1, 1]$, a contradiction to $r \in \mathcal{P}_{n-1}$. $\square$

Changing degree n in (2.6) to $n+1$, we obtain the following statement.

Corollary 2.6 For $\Delta = (x_i)_{i=0}^n \subset [-1, 1]$, let $\omega_\Delta(x) := \prod_{i=0}^n (x - x_i) \in \mathcal{P}_{n+1}$. Then, for all n , we have

$$\inf_{\Delta} \|\omega_\Delta\|_\infty = \|\omega_{\Delta_*}\|_\infty = 1/2^n. \tag{2.7}$$

Now, from (2.3) and (2.7), we get the smallest possible estimate for the error of interpolation.

Theorem 2.7 For $f \in C^{n+1}[-1, 1]$, the best choice of interpolating points (for the interpolant of degree n) is zeros of the Chebyshev polynomial T_{n+1} , namely $\Delta_* = (x_i^*) = (\cos \frac{2i+1}{2n+2} \pi)_{i=0}^n$, and we have

$$\|f - p_{\Delta_*}\|_\infty \leq \frac{1}{2^n} \frac{1}{(n+1)!} \|f^{(n+1)}\|_\infty$$

Example 2.8 For $f(x) = e^x$, and $x \in [-1, 1]$, the error of approximation provided by interpolating polynomial of degree 9 with 10 Chebyshev knots is bounded by

$$|e^x - p_9(x)| \leq \frac{1}{2^9} \frac{1}{10!} e \leq 1.5 \cdot 10^{-9}.$$