Probability Solutions (HOMEWORK, DECEMBER 2007)

Note Title 13/05/2008

1.

(a) $\psi_X(\theta) = \mathbb{E}\, e^{iX\theta}$.

(b) $\psi_{X+Y}(\theta) = \mathbb{E}\, e^{i(X+Y)\theta}$

$= \mathbb{E}\, e^{iX\theta} \cdot e^{iY\theta}$

$= \mathbb{E}\, e^{iX\theta} \cdot \mathbb{E}\, e^{iY\theta}$

(INDEPENDENCE OF X & Y)

$= \psi_X(\theta)\, \psi_Y(\theta)$.

(c) If $X_k \sim L(\mu_k, c_k; \alpha)$, $k = 1, 2$, then

$$\psi_{X_k}(\theta) = e^{i\mu_k\theta - c_k|\theta|^\alpha},$$

which implies (by (b))

$$\psi_{X_1+X_2}(\theta) = \psi_{X_1}(\theta)\, \psi_{X_2}(\theta)$$

$$= e^{i(\mu_1+\mu_2)\theta - (c_1+c_2)|\theta|^\alpha}.$$

Hence $X_1 + X_2 \sim L(\mu_1+\mu_2, c_1+c_2; \alpha)$.

(d) If $X_i \sim L(0, 1; \alpha)$, then

$$\psi_{X_i}(\theta) = e^{-|\theta|^\alpha}, \quad \text{for all } i.$$

Then

$$\psi_{\frac{X_1+\cdots+X_n}{n^{1/\alpha}}}(\theta) = \psi_{X_1+\cdots+X_n}(\theta/n^{1/\alpha})$$

$$= \prod_{k=1}^{n} \psi_{X_k}(\theta/n^{1/\alpha})$$

$$= \prod_{k=1}^{n} \psi_{X_k}\left(\frac{\theta}{n^{1/\alpha}}\right)$$

$$= \prod_{k=1}^{n} e^{-\left|\frac{\theta}{n^{1/\alpha}}\right|^{\alpha}}$$

$$= \prod_{k=1}^{n} e^{-\frac{|\theta|^{\alpha}}{n}}$$

$$= e^{-n \cdot \frac{|\theta|^{\alpha}}{n}} = e^{-|\theta|^{\alpha}}.$$

Hence $\dfrac{X_1 + \cdots + X_n}{n^{1/\alpha}} \sim L(0, 1; \alpha)$.

(e) If $X_i \sim L(\mu, c; \alpha)$, then

$$\psi_{A_n}(\theta) = \psi_{X_1 + \cdots + X_n}\left(\frac{\theta}{n}\right) = \prod_{k=1}^{n} \psi_{X_k}\left(\frac{\theta}{n}\right)$$

$$= \prod_{k=1}^{n} e^{i\mu\left(\frac{\theta}{n}\right) - c\left|\frac{\theta}{n}\right|^{\alpha}}$$

$$= e^{in\mu\left(\frac{\theta}{n}\right) - nc\left|\frac{\theta}{n}\right|^{\alpha}}$$

$$= e^{i\mu\theta - c|\theta|^{\alpha} n^{1-\alpha}}.$$

If $\alpha > 1$, then $n^{1-\alpha} \longrightarrow 0$, as $n \to \infty$.

Hence $\lim_{n \to \infty} \psi_{A_n}(\theta) = e^{i\mu\theta}$. This implies that the variance of A_n converges to zero and the mean converges to μ.

If $\alpha = 1$, then $\varphi_{A_n}(\theta) = e^{i\mu\theta - c|\theta|^\alpha}$,

for all n, i.e. $A_n \sim L(\mu, c; \alpha)$, for all n.

Thus no averaging behaviour is seen, as $n \to \infty$ — this is the Cauchy distribution.

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(a) $\mathbb{E} X_k = 1 . t + 0 . (1-t) = t$,

$$\text{Var } X_k = \mathbb{E}(X_k^2) - (\mathbb{E} X_k)^2$$

$$= 1.t + 0.(1-t) - t^2$$

$$= t(1-t).$$

(b) $$\mathbb{E} A_n = \frac{\mathbb{E} X_1 + \cdots + \mathbb{E} X_n}{n} = t.$$

$$\text{Var } A_n = \frac{\text{Var } X_1 + \cdots + \text{Var } X_n}{n^2} = \frac{n t(1-t)}{n^2}$$

$$= \frac{t(1-t)}{n}.$$

(c) We have (Chebyshev's Inequality), using (b),

$$\delta^2 \, \mathbb{P}(|A_n - t| > \delta) \le \operatorname{Var} A_n = \frac{t(1-t)}{n},$$

or

$$\mathbb{P}(|A_n - t| > \delta) \le \frac{t(1-t)}{n\delta^2}.$$

(d) $X_1 + X_2 + \cdots + X_n$ has the Binomial distribution with parameter t, i.e.

$$\mathbb{P}(X_1 + \cdots + X_n = k) = \binom{n}{k} t^k (1-t)^{n-k},$$

for $k = 0, 1, \ldots, n$. Hence

$$B_n f(t) = \mathbb{E} f\left(\frac{X_1 + \cdots + X_n}{n}\right)$$

$$= \sum_{k=0}^{n} f(k/n) \, \mathbb{P}(X_1 + \cdots + X_n = k)$$

$$= \sum_{k=0}^{n} f(k/n) \binom{n}{k} t^k (1-t)^{n-k}.$$

(e) $$|f(t) - B_n f(t)| = |f(t) - \mathbb{E} f(A_n)|$$

$$= \left| \mathbb{E}\left[f(t) - f(A_n) \right] \right|$$

$$\le \mathbb{E} |f(t) - f(A_n)|;$$

using $|\mathbb{E} U| \le \mathbb{E} |U|$.

(f) Given any $\epsilon > 0$, there exists $\delta > 0$ for which

$$|f(x) - f(y)| \leq \epsilon$$

for any $x, y \in [0,1]$ satisfying $|x-y| \leq \delta$.

Either $|A_n - t| \leq \delta$ or $|A_n - t| > \delta$.

Now $|A_n - t| \leq \delta$ implies

$$|f(A_n) - f(t)| \leq \epsilon.$$

Further, we also know that

$$|f(A_n) - f(t)| \leq |f(A_n)| + |f(t)|$$
$$\leq 2M,$$

for any A_n. Hence

$$|f(t) - B_n f(t)| \leq \mathbb{E}\, |f(t) - f(A_n)|$$
$$= \mathbb{E}\Big(|f(t) - f(A_n)| \,\Big|\, |A_n - t| \leq \delta\Big)$$
$$+ \mathbb{E}\Big(|f(t) - f(A_n)| \,\Big|\, |A_n - t| > \delta\Big)$$
$$\leq \epsilon\, \mathbb{P}(|A_n - t| \leq \delta) + 2M\, \mathbb{P}(|A_n - t| > \delta).$$

(g) Thus, by (f),

$$|f(t) - B_n f(t)| \leq \epsilon + 2M\,\mathbb{P}(|A_n - t| > \delta)$$

$$\leq \epsilon + \frac{2M t(1-t)}{n\delta^2},$$

by (c).

(h) First note that $t(1-t) \leq \frac{1}{4}$, for all $t \in [0,1]$

(The maximum occurs when $0 = \frac{d}{dt} t(1-t) = 1 - 2t$.)

Hence (g) implies

$$|f(t) - B_n f(t)| \leq \epsilon + \frac{M}{2n\delta^2}, \text{ for all } t \in [0,1].$$

If n is sufficiently enormous, then $\frac{M}{2n\delta^2} \leq \epsilon$. More precisely, if $n \geq \frac{M}{2\delta^2\epsilon}$, then $\frac{M}{2n\delta^2} \leq \epsilon$ and

$$|f(t) - B_n f(t)| \leq 2\epsilon, \text{ for any } t \in [0,1].$$

3.

(a) If $A \sim N(\mu, \sigma^2)$, then $\frac{A-\mu}{\sigma} \sim N(0,1)$, i.e.

$A = \mu + \sigma Z$, where $Z \sim N(0,1)$. Then

$$\mathbb{P}(A \geq \mu + T\sigma) = \mathbb{P}(\mu + \sigma Z \geq \mu + T\sigma)$$

$$= \mathbb{P}(Z \geq T) = 1 - \Phi(T).$$

Similarly, $B = \mu + (1+\delta)\sigma W$, where $W \sim N(0,1)$.

Hence

$$\mathbb{P}(B \geqslant \mu + T\sigma) = \mathbb{P}(\mu + (1+\delta)\sigma W \geqslant \mu + T\sigma)$$

$$= \mathbb{P}\left(W \geqslant \frac{T}{1+\delta}\right)$$

$$= 1 - \Phi\left(\frac{T}{1+\delta}\right).$$

(b) For Group A, we expect the number N_A of scholarships to roughly equal

$$N_A = 10000\, \mathbb{P}(A \geqslant \mu + T\sigma)$$

$$= 10000\,(1 - \Phi(2.5))$$

$$\simeq 62,$$

whilst for B,

$$N_B = 10000\left(1 - \Phi\left(\frac{2.5}{1.1}\right)\right)$$

$$\simeq 130.$$

Thus the observed frequencies are as expected.

For star scholarships,

$$N_A^* = 10000 \left(1 - \Phi(3) \right) \simeq 13$$

and

$$N_B^* = 10000 \left(1 - \Phi\left(\frac{3}{1\cdot 1}\right) \right) \simeq 32.$$

Again, they're roughly as expected.

(c) The key point is that scholarship / failure corresponds to $T \pm \sigma$, so the symmetry of the Normal distribution implies that the number of scholars / failures should be roughly equal. Hence Group B contains proportionately more stars / dunces than Group A, despite their equal means.

(d)
$$\int_x^\infty e^{-t^2/2}\,dt = \int_x^\infty \frac{1}{t} \cdot \underbrace{t e^{-t^2/2}}_{= \frac{d}{dt}\left(-e^{-t^2/2}\right)} dt$$

$$= \left[-t^{-1} e^{-t^2/2} \right]_{t=x}^{\infty} - \int_x^\infty \frac{d}{dt}\left(\frac{1}{t}\right)\left(-e^{-t^2/2}\right) dt$$

$$= x^{-1} e^{-x^2/2} - \int_x^\infty t^{-2} e^{-t^2/2}\,dt$$

$$= x^{-1} e^{-x^2/2} - R(x).$$

We've just shown that

$$\frac{\int_x^\infty e^{-t^2/2}\,dt}{x^{-1} e^{-x^2/2}} = 1 - \frac{R(x)}{x^{-1} e^{-x^2/2}},$$

i.e.,

$$\frac{1 - \Phi(x)}{(2\pi)^{-1/2}\, x^{-1} e^{-x^2/2}} = 1 - \frac{R(x)}{x^{-1} e^{-x^2/2}}.$$

We now show that

$$\lim_{x \to \infty} \frac{R(x)}{x^{-1} e^{-x^2/2}} = 0.$$

Now

$$\frac{R(x)}{x^{-1} e^{-x^2/2}} = x e^{x^2/2} \int_x^\infty t^{-2} e^{-t^2/2}\,dt$$

$$\underset{(t = x+s)}{=} x e^{x^2/2} \int_0^\infty (x+s)^{-2} e^{-(x+s)^2/2}\,ds.$$

Now $e^{x^2/2 - (x+s)^2/2} = e^{-xs} e^{-s^2/2} \le e^{-s^2/2}$, since $x, s \geq 0$ imply $e^{-xs} \le 1$. Further,

$$\frac{x}{(x+s)^2} \le \frac{1}{x}, \quad \text{for } s \geq 0.$$

Therefore,

$$\frac{R(x)}{x^{-1} e^{-x^2/2}} \le \frac{1}{x} \int_0^\infty e^{-s^2/2}\,ds = \frac{\sqrt{2/\pi}}{x},$$

which tends to zero, as $x \to \infty$.

(e) We have

$$\frac{\mathbb{P}(\text{B-pupil wins})}{\mathbb{P}(\text{A-pupil wins})} = \frac{1-\Phi\left(\frac{T}{1+\delta}\right)}{1-\Phi(T)} \quad (\text{by (a)})$$

$$= \frac{\int_{\frac{T}{1+\delta}}^{\infty} e^{-t^2/2}\,dt}{\int_{T}^{\infty} e^{-t^2/2}\,dt}$$

$$\simeq \frac{\left(\frac{T}{1+\delta}\right)^{-1} e^{-\left(\frac{T}{1+\delta}\right)^2/2}}{T^{-1}\, e^{-T^2/2}} \quad (\text{using (d)})$$

$$= (1+\delta)\, e^{-\frac{T^2}{2}\left\{\frac{1}{(1+\delta)^2}-1\right\}}$$

Now $\delta > 0$ implies $\frac{1}{(1+\delta)^2} - 1 > 0$. Thus

$$\lim_{T\to\infty} \frac{\mathbb{P}(\text{B wins})}{\mathbb{P}(\text{A wins})} = 0.$$