Let's begin with the integral

$$I_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} \cos^{2n}\theta \, d\theta$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} \left(\frac{e^{i\theta}+e^{-i\theta}}{2}\right)^{2n} d\theta$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} 2^{-2n}\sum_{k=0}^{2n}\binom{2n}{k} e^{ik\theta} e^{-i(2n-k)\theta} \, d\theta$$

$$= 2^{-2n}\binom{2n}{n}. \qquad (B1)$$

Now

$$\sqrt{n}\, I_n = \frac{\sqrt{n}}{\pi}\int_{-\pi/2}^{\pi/2} \cos^{2n}\theta \, d\theta$$

$$\underset{(\theta = t/\sqrt{n})}{=} \frac{1}{\pi}\int_{-\frac{\pi}{2}\sqrt{n}}^{\frac{\pi}{2}\sqrt{n}} \cos^{2n}(t/\sqrt{n})\, dt$$

$$\underset{(DCT)^*}{\longrightarrow} \frac{1}{\pi}\int_{-\infty}^{\infty} e^{-t^2}\, dt = \frac{1}{\sqrt{\pi}}! \qquad (B2)$$

Comparing (B1) and (B2), we obtain

$$\lim_{n\to\infty} \sqrt{n}\binom{2n}{n} 4^{-n} = \frac{1}{\sqrt{\pi}}, \qquad (B3)$$

or

$$\binom{2n}{n} 2^{-2n} \sim \frac{1}{\sqrt{\pi n}}. \qquad (B4).$$

* See the end of the note for a detailed discussion.

We can also check our calculation using Stirling's asymptotic formula for $n!$:

$$\lim_{n\to\infty} \frac{n!}{\sqrt{2\pi n}\, n^n e^{-n}} = 1$$

or

$$n! \sim \sqrt{2\pi n}\, n^n e^{-n}.$$

Then

$$\binom{2n}{n} \sim \frac{\sqrt{2\pi \cdot 2n}\,(2n)^{2n} e^{-2n}}{\left(\sqrt{2\pi n}\, n^n e^{-n}\right)^2}$$

$$= \frac{\sqrt{4\pi n}\; 2^{2n}\, \cancel{n^{2n}}\, \cancel{e^{-2n}}}{2\pi n \cdot \cancel{n^{2n}}\, \cancel{e^{-2n}}}$$

$$= \frac{\sqrt{2}}{\sqrt{2\pi n}} = \frac{1}{\sqrt{\pi n}}. \qquad (B5)$$

The Central Limit Theorem I

De Moivre & Laplace noticed that the central binomial coefficients $\left\{\binom{2m}{m+k} : -L \le k \le L\right\}$, where m is very big and L is small compared with m, seemed to fit a bell-shaped curve. Their analysis was similar to my earlier description of the birthday problem.

Their key insight was to consider the ratio

$$p_{mk} = \frac{\binom{2m}{m+k}}{\binom{2m}{m}} = \frac{(2m)!}{(m-k)!\,(m+k)!} \cdot \frac{(m!)^2}{(2m)!}$$

$$= \frac{(m!)^2}{(m-k)!\,(m+k)!}$$

$$= \frac{m!\,(m(m-1)\cdots(m-k+1))(m-k)!}{(m-k)!\,(m+k)(m+k-1)\cdots(m+1)m!}$$

$$= \frac{m(m-1)\cdots(m-k+1)}{(m+k)(m+k-1)\cdots(m+1)}$$

dividing top & bottom by m

$$= \frac{(1-\frac{1}{m})(1-\frac{2}{m})\cdots(1-\frac{k-1}{m})}{(1+\frac{1}{m})(1+\frac{2}{m})\cdots(1+\frac{k}{m})}.$$

This should remind you of the birthday problem. Taking logarithms,

$$\ln p_{mk} = \sum_{j=1}^{k-1} \ln\left(1-\frac{j}{m}\right) - \sum_{j=1}^{k} \ln\left(1+\frac{j}{m}\right)$$

$$\simeq -\sum_{j=1}^{k-1} \frac{j}{m} - \sum_{j=1}^{k} \frac{j}{m} = \frac{-k(k-1)-(k+1)k}{2m} = -\frac{k^2}{m}.$$

Hence

$$\ln p_{mk} \simeq e^{-k^2/m},$$

and, if $k = t\sqrt{m}$, then $\ln p_{m\,t\sqrt{m}} \simeq e^{-t^2}$.

The Central Limit Theorem III

We use

$$\binom{2n}{n+2k} 2^{-n} = \frac{1}{2\pi} \int_{-\pi}^{\pi} \cos^{2n}\theta \; e^{-2ik\theta} \, d\theta$$

$$= \frac{1}{\pi} \int_{-\pi/2}^{\pi/2} \cos^{2n}\theta \; e^{-2ik\theta} \, d\theta$$

$$\underset{\left(\theta = \frac{t}{\sqrt{n}}\right)}{=} \frac{1}{\pi} \int_{-\frac{\pi}{2}\sqrt{n}}^{\frac{\pi}{2}\sqrt{n}} \cos^{2n}\left(\frac{t}{\sqrt{n}}\right) e^{-2ikt/\sqrt{n}} \, \frac{dt}{\sqrt{n}}.$$

(B6)

Thus

$$\sqrt{n}\binom{2n}{n+2k} 2^{-2n} = \frac{1}{\pi}\int_{-\frac{\pi}{2}\sqrt{n}}^{\frac{\pi}{2}\sqrt{n}} \cos^{2n}\left(\frac{t}{\sqrt{n}}\right) e^{-2ikt/\sqrt{n}}\, dt \qquad (B7),$$

or

$$\frac{\sqrt{n}\,\Gamma(2n+1)\, 2^{-2n}}{\Gamma(n+2k+1)\,\Gamma(n-2k+1)} = \frac{1}{\pi}\int_{-\frac{\pi}{2}\sqrt{n}}^{\frac{\pi}{2}\sqrt{n}} \cos^{2n}\left(\frac{t}{\sqrt{n}}\right) e^{-2ikt/\sqrt{n}}\, dt \qquad (B7'),$$

which emphasizes that both sides analytically continue. Now, writing $2k = x\sqrt{n}$, x fixed, we obtain

$$\frac{\sqrt{n}\,\Gamma(2n+1)\, 2^{-2n}}{\Gamma(n+x\sqrt{n}+1)\,\Gamma(n-x\sqrt{n}+1)} = \frac{1}{\pi}\int_{-\frac{\pi}{2}\sqrt{n}}^{\frac{\pi}{2}\sqrt{n}} \cos^{2n}\left(\frac{t}{\sqrt{n}}\right) e^{-ixt}\, dt$$

$$\xrightarrow[(DCT)]{} \frac{1}{\pi}\int_{\mathbb{R}} e^{-t^2/2}\, e^{-ixt}\, dt = \frac{e^{-x^2/2}\sqrt{2\pi}}{\pi}$$

$$= \sqrt{\frac{2}{\pi}}\; e^{-x^2/2}, \qquad (B8)$$

or

$$\lim_{n\to\infty} \sqrt{n}\binom{2n}{n+x\sqrt{n}} 2^{-2n} = \sqrt{\frac{2}{\pi}}\; e^{-x^2/2}. \qquad (B9)$$

This is a special case of the central limit theorem.

Bernoulli Trials

We say that X is a BERNOULLI RANDOM VARIABLE if X can only take two values. For example, if $\mathbb{P}(X=1) = p$ and $\mathbb{P}(X=0) = 1-p$.

A sequence of Bernoulli trials is then just a sequence of iid Bernoulli random variables $X_1, X_2, X_3, \ldots$; in other words, this is the mathematical idealization of tossing a coin infinitely many times. We let

$$S_n = X_1 + \cdots + X_n$$

and

$$A_n = \frac{X_1 + \cdots + X_n}{n}.$$

Then $\mathbb{E} X_i = p \cdot 1 + (1-p) \cdot 0 = p$
and

$$\mathbb{E}(X_i^2) = p \cdot 1 = p,$$

which implies

$$\begin{aligned} \operatorname{Var} X_i &= \mathbb{E}(X_i^2) - (\mathbb{E} X_i)^2 \\ &= p - p^2 = p(1-p). \end{aligned}$$

Then

$$\mathbb{E}\, A_n = p.$$

Now

$$\mathbb{E}\,(A_n - \mathbb{E} A_n)^2 = \mathbb{E}\left(\frac{X_1 + \cdots + X_n}{n} - p\right)^2$$

$$= \mathbb{E}\left(\frac{X_1 - p + X_2 - p + \cdots + X_n - p}{n}\right)^2$$

$$= \frac{1}{n^2}\left[\mathbb{E}\,(X_1 - p)^2 + \cdots + \mathbb{E}\,(X_n - p)^2\right]$$

$$= \frac{1}{n^2}\left(\operatorname{Var} X_1 + \cdots + \operatorname{Var} X_n\right)$$

$$= \frac{p(1-p)}{n} \equiv \frac{\sigma^2}{n}.$$

Hence $\{A_n\}$ is a sequence of random variables for which $\mathbb{E} A_n = p$ and $\operatorname{Var} A_n = \frac{\sigma^2}{n} \longrightarrow 0$, as $n \longrightarrow \infty$.

Chebyshev's Inequality Let X be any random variable with $\mathbb{E}X = \mu$, $\operatorname{Var} X = \sigma^2$. Then

$$\mathbb{P}\left(|X - \mu| \geq \delta \right) \leq \frac{\sigma^2}{\delta^2}.$$

Proof:

$$\sigma^2 = \mathbb{E}(X-\mu)^2 \geq \mathbb{P}\left(|X-\mu| \geq \delta \right) \delta^2. \qquad \square$$

Weak Law of Large Numbers: For any $\delta > 0$,

$$\mathbb{P}\left(|A_n - p| \geq \delta \right) \leq \frac{\sigma^2}{n\delta^2}.$$

Proof:

$$\frac{\sigma^2}{n} = \mathbb{E}(A_n - p)^2 \geq \delta^2 \, \mathbb{P}\left(|A_n - \delta| \geq \delta \right).$$

$\square$

BERNSTEIN POLYNOMIALS

Define
$$B_n f(p) = \mathbb{E}\, f(A_n)$$
$$= \sum_{k=0}^{n} f\left(\frac{k}{n}\right) \binom{n}{k} p^k (1-p)^{n-k}.$$

Then $B_n f$ is a polynomial of degree n. Since A_n is likely to be close to $f(p)$. In fact much more is true. First observe that

$$|f(p) - B_n f(p)| = \left|\mathbb{E}\left(f(p) - f(A_n)\right)\right|.$$
$$\leq \mathbb{E}\, |f(p) - f(A_n)|$$

Now suppose that $f : [0,1] \longrightarrow \mathbb{R}$ is a continuous function. A theorem in analysis states that any such f is UNIFORMLY CONTINUOUS : given any $\epsilon > 0$, there is a $\delta > 0$ for which $|x-y| \leq \delta$ implies $|f(x) - f(y)| \leq \epsilon$, for any $x, y \in [0, 1]$.

Then

$$|f(p) - B_n f(p)|$$

$$\leq 2\|f\|_\infty \, \mathbb{P}(|A_n - p| \geq \delta)$$

$$+ \epsilon \, \mathbb{P}(|A_n - p| < \delta)$$

$$\leq \frac{2\|f\|_\infty \sigma^2}{n\delta} + \epsilon .$$

Hence, given $\epsilon > 0$, choosing an enormous n implies

$$|f(p) - B_n f(p)| \leq 2\epsilon$$

for any $p \in [0, 1]$.