PROBLEMS IN MATHEMATICS: NEWTON AND POLYNOMIAL INTERPOLATION

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Abstract. These notes provide a short introduction to divided differences and the Newton form of the interpolating polynomial for Problems in Mathematics 2024. Those students who need to complete the Exercises will find them in Section 4. Students interested in an essay on this topic will find suggestions in Section 5.

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1. Introduction

You can access these notes, and other material, via my office machine:
`http://econ109.econ.bbk.ac.uk/brad/Problems_in_Maths/`

Some students might also be interested in the software packages Matlab and Octave, mentioned in the lecture.
`http://econ109.econ.bbk.ac.uk/brad/Methods/matlab_intro_notes.pdf`

You can find out more about Numerical Analysis in my lecture notes:
`http://econ109.econ.bbk.ac.uk/brad/teaching/Methods/nabook.pdf`

2. Historical Introduction

We begin roughly 400 years ago, at the beginning of the Seventeenth Century. At that time, applied mathematics was expanding rapidly, from providing better tables of trigonometric functions for European oceanic navies, to improved ways to calculate interest on debts. This might sound mundane, but war and money have always been closely linked to mathematical applications. Algebra was a Sixteenth century invention, but almost nothing was known of calculus in 1600. Fortunately, this was about to change!

To gain some idea of the needs of the time we need to understand the recent history of computation. My phone can compute 10^8 FLOPS (floating point operations per second) and cost £100, but in 1990 a research workstation would be limited to 10^6 FLOPS and cost £10^4. Going back to the 1970s, schoolchildren were still taught how to use trigonometric and logarithmic tables, which would have looked something like this (x is measured in degrees):

x	$\cos x$
0	1.0000
1	0.99985
2	0.99939
3	0.99863
4	0.99756
5	0.99619

Such four-figure tables required enormous work to produce in 1700, but what do we do if we need to calculate $\cos 1.4$? One simple way is linear interpolation using the table: we let $f(x) = \cos x$ and define

$$p(x) = f(1) + (f(2) - f(1))(x - 1), \tag{1}$$

Using (1), we obtain

$$p(1.4) = 0.99966.$$

The true value is $\cos 1.4 = 0.9997014897811831\ldots$, so this isn't too bad: an error of roughly 3.65×10^{-5}. Can we do better?

One idea is to use three values and fit a quadratic. Let $a = 1$, $b = 2$ and $c = 0$ and let's write the quadratic in the form

$$q(x) = p(x) + Q(x - a)(x - b), \tag{2}$$

where $p(x)$ is given by (1). The idea here is to avoid further computation, since we already know that $p(a) = \cos a$ and $p(b) = \cos b$. Since the quadratic term $(x - a)(x - b)$ vanishes at a and b, the quadratic $q(x)$ already satisfies $q(a) = \cos a$ and $q(b) = \cos b$. To reproduce the value at c, we solve

$$\cos c = p(c) + Q(c - a)(c - b),$$

or

$$Q = \frac{\cos c - p(c)}{(c - a)(c - b)}.$$

The new approximation is

$$q(1.4) = 0.9997014954967155$$

which is *much* closer to the true figure

$$\cos 1.4 = 0.9997014897811831$$

since the error is now roughly 5.71×10^{-9}. Thus moving from linear interpolation to quadratic interpolation has reduced our error by 10^4, which is rather good!

3. Polynomial Interpolation

Let

$$z_0, z_1, \ldots, z_n$$

be any different complex numbers and let

$$f_0, \ldots, f_n$$

be any complex numbers (they don't need to be distinct). We want to construct a polynomial p of degree n for which

$$p(z_j) = f_j, \qquad \text{for } 0 \le j \le n.$$

Such a polynomial is called an **interpolating polynomial**, and we say that p **interpolates** the data

$$(z_0, f_0), (z_1, f_1), \ldots, (z_n, f_n).$$

We shall let $\mathbb{P}_n$ denote the vector space of polynomials of degree at most n.

Example 3.1. *How do we find the quadratic polynomial satisfying $p(0) = \alpha$, $p(1) = \beta$ and $p(4) = \gamma$? We could just substitute $p(x) = p_0 + p_1 x + p_2 x^2$ and solve the three linear equations to obtain the coefficients. However, there is a simpler solution: we write*

$$p(z) = \alpha \frac{(z-1)(z-4)}{(0-1)(0-4)} + \beta \frac{z(z-4)}{(1-0)(1-4)} + \gamma \frac{z(z-1)}{(4-0)(4-1)}.$$

The key point in Example 3.1 is that the polynomial

$$\ell_0(z) = \frac{(z-1)(z-4)}{(0-1)(0-4)} = \frac{1}{4}(z-1)(z-4)$$

satisfies

$$\ell_0(1) = \ell_0(4) = 0 \qquad \text{and} \qquad \ell_0(0) = 1.$$

Specifically, if we let $z_0 = 0$, $z_1 = 1$ and $z_2 = 4$, then we have

$$\ell_0(z_k) = \begin{cases} 1 & \text{if } k = 0, \\ 0 & \text{if } k \ne 0. \end{cases}$$

Thus

$$\ell_1(z) = \frac{z(z-4)}{(1-0)(1-4)} = -\frac{1}{3} z(z-4) \quad \text{satisfies} \quad \ell_1(z_k) = \begin{cases} 1 & \text{if } k = 1, \\ 0 & \text{if } k \ne 1, \end{cases}$$

and

$$\ell_2(z) = \frac{z(z-1)}{(4-0)(4-1)} = \frac{1}{12} z(z-1) \quad \text{satisfies} \quad \ell_2(z_k) = \begin{cases} 1 & \text{if } k = 2, \\ 0 & \text{if } k \ne 2. \end{cases}$$

Example 3.2. *Suppose now that $\alpha = \beta = \gamma = 1$ in Example 3.1. We have shown that*

$$\begin{aligned} 1 &= \ell_0(z) + \ell_1(z) + \ell_2(z) \\ &= \frac{1}{4}(z-1)(z-4) + \frac{-1}{3} z(z-4) + \frac{1}{12} z(z-1). \end{aligned}$$

If we now divide both sides by $z(z-1)(z-4)$, then we obtain

$$\frac{1}{z(z-1)(z-4)} = \frac{(1/4)}{z} + \frac{(-1/3)}{z-1} + \frac{(1/12)}{z-4}.$$

We can now generalise the trick used in Example 3.1.

Lemma 3.1. *Let*

$$\ell_j(z) = \frac{(z-z_0)(z-z_1)\cdots(z-z_{j-1})(z-z_{j+1})\cdots(z-z_n)}{(z_j-z_0)(z_j-z_1)\cdots(z_j-z_{j-1})(z_j-z_{j+1})\cdots(z_j-z_n)},$$

or, more briefly,

$$\ell_j(z) = \prod_{k=0, k\neq j}^{n} \frac{z-z_k}{z_j-z_k}, \qquad \text{for } 0 \le j \le n. \tag{3}$$

Then $\ell_j \in \mathbb{P}_n$ *and*

$$\ell_j(z_i) = \begin{cases} 1 & \text{if } i = j, \\ 0 & \text{if } i \neq j, \end{cases}$$

for $0 \le i, j, \le n$.

Proof. By construction, $\ell_j(z_i) = 0$ when $i \neq j$, because the product in (3) contains the term $(z - z_i)$. However, $\ell_j(z_j) = 1$, because then every term in (3) occurs in both numerator and denominator. □

These polynomials $\ell_0, \ell_1, \ldots, \ell_n$ are useful because they allow us to write down a very simple expression for the polynomial interpolant.

Proposition 3.2. *The interpolating polynomial* $p \in \mathbb{P}_n$ *for the data* $\{(z_j, f_j) : 0 \le j \le n\}$ *is given by*

$$p(z) = \sum_{j=0}^{n} f_j \ell_j(z), \qquad z \in \mathbb{C}. \tag{4}$$

Proof. Equation 3 implies $p(z_k) = \sum_{j=0}^{n} f_j \ell_j(z_k) = f_k$, $0 \le k \le n$. □

Equation 4 is called the **Lagrange form of the interpolating polynomial**. It's extremely useful in theoretical work and there are new applications in modern numerical approximation.

Uniqueness requires a simple lemma.

Lemma 3.3. *Let* $p(z) = a_0 + a_1 z + a_2 z^2 + \cdots + a_n z^n$, *where* $z \in \mathbb{C}$ *and* $a_0, a_1, \ldots, a_n \in \mathbb{C}$. *Then* $p(z)$ *has at most* n *distinct zeros in* $\mathbb{C}$ *unless* $a_0 = a_1 = \cdots = a_n = 0$.

Proof. The lemma is plainly true when $n = 0$ or $n = 1$. We then proceed by induction. Thus let us assume that every polynomial of degree less than n has at most n different zeros, unless every coefficient is zero. Given any polynomial $p(z)$ of degree $n + 1$, **either** p has a root, say $p(w) = 0$, **or** p has no roots. If the latter condition is valid, then there's nothing further to demonstrate. If the former is valid, then $(z - w)$ is a factor of $p(z)$. Thus we can write $p(z) = q(z)(z - w)$, and the roots of p are w and the roots of q. However, by induction hypothesis, q can have at most n different roots. Thus p can have, in total, at most $n + 1$ different roots. □

The last lemma is a very simple version of the great Fundamental Theorem of Algebra: a polynomial of degree n with complex coefficients has exactly n complex zeros if we count multiple zeros multiply. (Thus $(z - 2)^2$ has two zeros.)

Proposition 3.4. *There is exactly one interpolating polynomial* $p \in \mathbb{P}_n$ *when the points* $z_0, z_1, \ldots, z_n$ *are distinct.*

Proof. Existence was shown in Proposition 3.2, so we address uniqueness. Thus let p and q be interpolating polynomials of degree n. Their difference $p - q$ is a polynomial of degree n that vanishes at the $n+1$ different points $z_0, \ldots, z_n$. Hence $p - q$ vanishes identically, using the last lemma. □

The Lagrange form of the interpolating polynomial is useful when n is small and in theoretical work. However, it is particularly inconvenient if we have constructed $p_{n-1} \in \mathbb{P}_{n-1}$ interpolating data $\{(z_j, f_j) : 0 \le j \le n-1\}$ and are then given a new datum (z_n, f_n), because we almost have to start the calculation from scratch. Fortunately a more compact form is available. The key idea is to let $p \in \mathbb{P}_n$ take the form

$$p_n(z) = p_{n-1}(z) + C(z - z_0)(z - z_1) \cdots (z - z_{n-1}), \qquad z \in \mathbb{C}. \tag{5}$$

We see that $p_n(z_j) = p_{n-1}(z_j) = f_j$, for $0 \le j \le n-1$, so we do not disturb our previous interpolant at these points. Of course we choose C to satisfy the equation

$$f_n = p_{n-1}(z_n) + C \prod_{k=0}^{n-1} (z_n - z_k). \tag{6}$$

Obviously C depends on f and $z_0, z_1, \ldots, z_n$. A traditional notation is

$$C = f[z_0, z_1, \ldots, z_n], \tag{7}$$

so that 5 becomes

$$p_n(z) = p_{n-1}(z) + f[z_0, z_1, \ldots, z_n](z - z_0)(z - z_1) \cdots (z - z_{n-1}). \tag{8}$$

The number $f[z_0, \ldots, z_n]$ is called a **divided difference**, because of the method used to calculate these numbers described below. Note that the coefficient of highest degree for p_n does not depend on the order in which we take the points. In other words, if we replace $z_0, z_1, \ldots, z_n$ by $z_{\pi 0}, z_{\pi 1}, \ldots, z_{\pi n}$, for any permutation π of the numbers $\{0, 1, \ldots, n\}$, then $f[z_{\pi 0}, \ldots, z_{\pi n}] = f[z_0, \ldots, z_n]$. Another way to see this is the following explicit expression for $f[z_0, \ldots, z_n]$, which is sometimes useful in theoretical work.

Proposition 3.5. *We have*

$$f[z_0, z_1, \ldots, z_n] = \sum_{j=0}^{n} \frac{f(z_j)}{\prod_{k=0, k \ne j}^{n} (z_j - z_k)}. \tag{9}$$

Further, $f[z_0, \ldots, z_n] = 0$ *when* f *is a polynomial of degree less than* n.

Proof. We just equate the coefficients of z^n in $p_n(z) = \sum_{j=0}^n f(z_j) \ell_j(z)$, using Proposition 3.2. Moreover, if $f(z) = z^\ell$ and $\ell < n$, then the coefficient of z^n in p_n is zero. But this highest degree coefficient is $f[z_0, \ldots, z_n]$. □

Recurring equation 8, and defining $f[z_0] = f(z_0)$, yields the explicit expression

$$\begin{aligned} p_n(z) = f[z_0] + f[z_0, z_1](z - z_0) + f[z_0, z_1, z_2](z - z_0)(z - z_1) + \cdots \\ + f[z_0, z_1, \ldots, z_n](z - z_0)(z - z_1) \cdots (z - z_{n-1}), \end{aligned}$$

and this is called the **Newton form** of the interpolating polynomial.

It's **important** to understand that $f[z_0, \ldots, z_\ell]$ is the coefficient of highest degree for the polynomial $p_\ell \in \mathbb{P}_\ell$ interpolating the data $\{(z_k, f_k) : 0 \le k \le \ell\}$.

Example 3.3. *The Newton form of the quadratic polynomial satisfying* $p(0) = f(0)$, $p(1) = f(1)$ *and* $p(4) = f(4)$ *is*

$$p(z) = f[0] + f[0, 1]z + f[0, 1, 4]z(z - 1).$$

You'll see how to calculate the coefficients shortly.

The recursion used to calculate divided difference and justifying the suitability of their name is derived in the following key theorem.

Theorem 3.6. *For any distinct complex numbers* $z_0, z_1, \ldots, z_n, z_{n+1}$ *the divided differences satisfy*

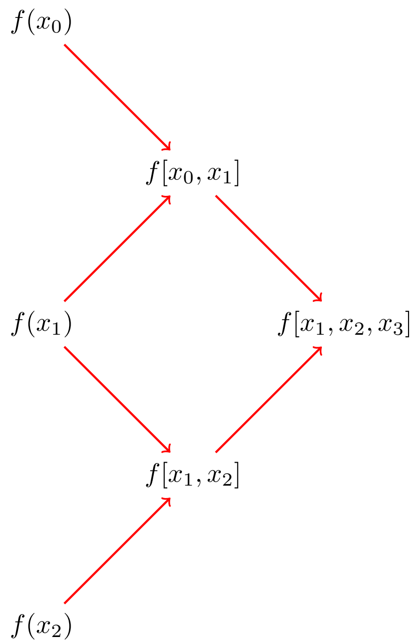
$$f[z_0, \ldots, z_{n+1}] = \frac{f[z_0, \ldots, z_n] - f[z_1, \ldots, z_{n+1}]}{z_0 - z_{n+1}}. \tag{10}$$

Proof. We introduce two polynomials: (i) $p \in \mathbb{P}_n$ interpolates $\{(z_k, f_k) : 0 \le k \le n\}$, and (ii) $q \in \mathbb{P}_n$ interpolates $\{(z_k, f_k) : 1 \le k \le n+1\}$. Thus the coefficients of highest degree for p and q are $f[z_0, \ldots, z_n]$ and $f[z_1, \ldots, z_{n+1}]$, respectively. The key **trick** is now the observation that the polynomial $r \in \mathbb{P}_{n+1}$ interpolating at all $n+1$ points satisfies

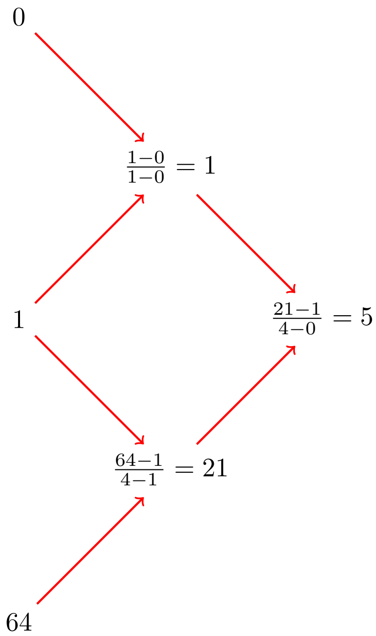
$$r(z) = \frac{(z - z_{n+1})p(z) - (z - z_0)q(z)}{z_0 - z_{n+1}}, \tag{11}$$

because it is unique, by Proposition 3.4, and it is easily checked that the right hand side of (11) interpolates at $z_0, \ldots, z_{n+1}$: an exercise for the reader. Now the coefficient of highest degree in r is $f[z_0, \ldots, z_{n+1}]$, so equating the coefficients of highest degree in (11) yields (10). □

Proposition 5 is extremely important. It's the basis of the algorithm for divided differences and the Newton form of the interpolating polynomial. One way to illustrate the divided difference table is as follows.



Example 3.4. *Let $f(x) = x^3$ and let $x_0 = 0$, $x_1 = 1$ and $x_2 = 4$. The divided difference table is as follows.*



Thus $f[x_0, x_1] = 1$, $f[x_0, x_1, x_2] = 5$ and the Newton form of the quadratic interpolating $f(x) = x^3$ at 0, 1 and 4 is given by

$$p(x) = x + 5x(x-1).$$

In linear algebra terms we form the lower triangular matrix

$$\begin{pmatrix} x_0 & f(x_0) & & & & & \\ x_1 & f(x_1) & f[x_0,x_1] & & & & \\ x_2 & f(x_2) & f[x_1,x_2] & f[x_0,x_1,x_2] & & & \\ \vdots & \vdots & \vdots & \vdots & \ddots & & \\ x_{n-1} & f(x_{n-1}) & f[x_{n-2},x_{n-1}] & f[x_{n-3},x_{n-2},x_{n-1}] & \ldots & f[x_0,x_1,\ldots,x_{n-1}] & \\ x_n & f(x_n) & f[x_{n-1},x_n] & f[x_{n-2},x_{n-1},x_n] & \ldots & f[x_1,x_2,\ldots,x_n] & f[x_0,\ldots,x_n] \end{pmatrix}.$$

In practice, only the first two columns of this lower triangular matrix are stored. The diagonal elements are those needed for the Newton form of the interpolating polynomial, so it's usual for the second column to be overwritten by subsequent columns; that's $\mathcal{O}(n)$ rather than $\mathcal{O}(n^2)$ numbers to be stored. At completion, the second column should contain the diagonal elements of the matrix, that is

$$[f(x_0), f[x_0,x_1], f[x_0,x_1,x_2], f[x_0,x_1,x_2,x_3], \ldots, f[x_0,x_1,\ldots,x_{n-1}], f[x_0,\ldots,x_n]\,]^T.$$

What about the error in polynomial interpolation?

Theorem 3.7. *Let $p \in \mathbb{P}_n$ interpolate f at n distinct complex numbers $z_0, z_1, \ldots, z_n$. Then the error $e = f - p$ satisfies the equation*

$$e(w) = f[z_0, z_1, \ldots, z_n, w] \prod_{k=0}^{n} (w - z_k), \qquad w \in \mathbb{C}. \tag{12}$$

Proof. If we add a new interpolation point z_{n+1}, then the Newton interpolating polynomial $q \in \mathbb{P}_{n+1}$ is given by

$$q(z) = p(z) + f[z_0, z_1, \ldots, z_n, z_{n+1}] \prod_{k=0}^{n} (z - z_k).$$

Hence

$$f(z_{n+1}) = p(z_{n+1}) + f[z_0, z_1, \ldots, z_n, z_{n+1}] \prod_{k=0}^{n} (z - z_k).$$

Since z_{n+1} can be *any* point, we can write $w = z_{n+1}$, which completes the proof. □

This result is of little use for error bounds unless we can bound $f[z_0, z_1, \ldots, z_n, w]$ from above in some way. Now the first mean value theorem implies the equation

$$f[x_0, x_1] = \frac{f(x_1) - f(x_0)}{x_1 - x_0} = f'(\alpha),$$

for some point $\alpha \in [x_0, x_1]$. There is an important result for divided differences generalising this remark that's essentially a form of the mean value theorem you'll meet in real analysis. We shall use this relation to express the error in terms of the maximum modulus of the $(n+1)$st derivative of f.

Theorem 3.8. *Let f have continuous $(n+1)$st derivative and let $x_0 < x_1 < \cdots < x_n$ be* **real** *numbers. Then there is a point $\alpha \in [x_0, x_n]$ such that*

$$f[x_0, x_1, \ldots, x_n] = \frac{f^n(\alpha)}{n!}. \tag{13}$$

Proof. Let $p_n \in \mathbb{P}_n$ interpolate f at $x_0, \ldots, x_n$. Then the error function $e = f - p_n$ has at least $n+1$ zeros in $[x_0, x_n]$. Hence its derivative e' has at least n zeros in $[x_0, x_n]$, and its second derivative e'' has at least $n-1$ zeros. Continuing in this way, we deduce that $e^{(n)}$ has at last one zero, α say, in $[x_0, x_n]$. But then

$$0 = e^{(n)}(\alpha) = f^{(n)}(\alpha) - f[x_0, \ldots, x_n]n!,$$

as required. □

Corollary 3.9. *Let f have continuous $(n+1)$st derivative and let $x_0, x_1, \ldots, x_n$ be different real numbers. If $p_n \in \mathbb{P}_n$ is the interpolating polynomial, then the error $e_n = f - p_n$ satisfies*

$$|e_n(x)| \le \frac{M \prod_{k=0}^{n} |x - x_k|}{(n+1)!}, \qquad x \in [a, b], \tag{14}$$

where $M = \max\{|f^{(n+1)}(t)| : a \le t \le b\}$.

Proof. This is immediate from the last two theorems. □

These examples might suggest that increasing the number of interpolation points always decreases the error. This is **not** so, as you may see in exercises.

Example 3.5. *Let $f(x) = \exp(x)$ and let $a = -1/2$, $b = 1/2$. If the interpolation points are always contained within the interval $[-1/2, 1/2]$, then the error of interpolation satisfies*

$$|e_n(x)| \le \frac{e}{(n+1)!}, \qquad -1/2 \le x \le 1/2.$$

In other words, the error is tiny, what ever the choice of interpolation points. In fact, this is true whenever the function being interpolated is complex differentiable at every point of the complex plane. It is certainly **not** *true for general functions (see the essay topic on the Runge phenomenon for more information).*

Equation (14) suggests the following problem: Find interpolation points $(x_k)_{k=0}^{n-1}$ minimising

$$\max_{-1 \le x \le 1} \left(\prod_{k=0}^{n-1} |x - x_k| \right), \tag{15}$$

which occurs when we want to minimise upper bound (14) on the interval $[-1, 1]$. Equally spaced points are particularly bad. In fact, the minimum value of (15) occurs when

$$\prod_{k=0}^{n-1} (x - x_k) = 2^{1-n} \cos(n \cos^{-1} x).$$

This was discovered by the great Russian mathematician Chebyshev and is called a Chebyshev polynomial.

4. Exercises

DEADLINE: 14:00 GMT on January 25, 2024. Please submit your scanned PDF via Moodle.

A total mark out of 40 will be given. This problem sheet is worth 10% of the marks for this module. Note: College regulations mean that, unless there are mitigating circumstances, work submitted late (up to 14 days) will have the mark capped at 40% (i.e. 16/40) and work submitted after 14 days late will score zero. Full coursework regulations are given in your handbook.

(1) Let $h > \alpha > 0$.

(a) Find the linear polynomial

$$p(x) = a_0 + a_1 x$$

interpolating f at $\pm\alpha$, i.e. find the coefficients a_0 and a_1. Show that

$$\int_{-h}^{h} p(x)\,dx = h\Big(f(-\alpha) + f(\alpha)\Big).$$

[**Hint**: The integral

$$\int_{-h}^{h} x^k\,dx$$

vanishes when k is an odd positive integer.]

(b) Now find the value of α for which

$$\int_{-h}^{h} f(x)\,dx = h\Big(f(-\alpha) + f(\alpha)\Big)$$

when f is any cubic polynomial. What happens if f is a quartic polynomial?

[**Hint:** Try $f(x) = x^2$, $f(x) = x^3$ and $f(x) = x^4$ and use linearity: if p interpolates f and q interpolates g at $x_0, x_1, \ldots, x_n$, then $Ap + Bq$ interpolates $Af + Bg$ at the same points.]

(2) Let n be any positive integer and let $\omega = \exp(2\pi i/n)$. You may use the facts that the complex numbers $1, \omega, \omega^2, \ldots, \omega^{n-1}$ form the vertices of a regular polygon on the unit circle

$$\{z \in \mathbb{C} : |z| = 1\}$$

in the complex plane and $z = \omega^k$ satisfies $z^n = 1$ for $k = 0, 1, \ldots, n-1$. [The somewhat archaic jargon is to say that the ω^k are the n^{th} **roots of unity**.]

(a) Show that

$$\ell_0(z) = \frac{1}{n}\left(\frac{z^n - 1}{z - 1}\right)$$

is the unique polynomial of degree $n-1$ for which

$$\ell_0(\omega^k) = \begin{cases} 1 & \text{if } k = 0, \\ 0 & \text{if } k \neq 0. \end{cases}$$

[**Hint:** Use de l'Hôpital's rule.]

(b) Prove that

$$\ell_0(z) = \prod_{k=1}^{n-1} \frac{z - \omega^k}{1 - \omega^k}.$$

(c) Prove that

$$\ell_j(z) = \ell_0(\omega^{-j} z), \qquad \text{for } j = 1, 2, \ldots, n-1,$$

satisfies

$$\ell_j(\omega^k) = \begin{cases} 1 & \text{if } k = j, \\ 0 & \text{if } k \neq j. \end{cases}$$

(d) Define

$$p(z) = \sum_{j=0}^{n-1} \omega^{-j} \ell_j(z).$$

Prove that $p(z) \equiv z^{n-1}$. [**Hint:** Show that both $p(z)$ and $q(z) = z^{n-1}$ interpolate $f(z) = 1/z$ at $1, \omega, \omega^2, \ldots, \omega^{n-1}$.]

(3) (a) Find the divided difference table for $f(x) = x^3$ at the five points $0, 1, 2, 3, 4$.

(b) Define the forward difference operator

$$\Delta f(x) = f(x+1) - f(x).$$

Show that

$$\Delta^2 f(x) = f(x+2) - 2f(x+1) + f(x).$$

(c) Define the **shift operator**

$$Ef(x) = f(x+1),$$

so that

$$\Delta f(x) = \Big(E - 1\Big) f(x).$$

Hence show that

$$\Delta^3 f(x) = (E-1)^3 f(x) = f(x+3) - 3f(x+2) + 3f(x+1) - f(x).$$

(d) Use the divided difference recurrence relation and induction to prove that

$$f[x, x+1, x+2, \ldots, x+n] = \frac{\Delta^n f(x)}{n!}$$

5. Essay Questions

In addition to the exercises, you must choose two essay questions to complete over the year, at most one per topic. Each essay is worth 30% towards the marks for the module. More details, including essay deadlines and guidelines for writing essays, are in the module information handout, which is available on Moodle. I can provide copies of all the books referenced to any interested student.

5.1. **Chebyshev Polynomials.** Write an essay on Chebyshev polynomials and Chebyshev approximation. You can find lots of interesting material in [5].

5.2. **Advanced Properties of Divided Differences.** Write an essay on more advanced properties of divided differences and exponential Brownian motion. This topic is only suitable for students who have attended my course *Mathematical and Numerical Methods* at MSc level. You can find most of the material required in Section 4 of my own paper "Functionals of Exponential Brownian Motion and Divided Differences", which is available on my server:

`http://www.cato.tzo.com/brad/papers`

5.3. **The Runge Phenomenon.** Write an essay on the Runge Phenomenon. You should include the following key points below, which should provide a fairly detailed basis for your essay.

The aim of this detailed introduction is to show that interpolating the function

$$f(z) = \frac{1}{\alpha - z}, \tag{16}$$

at equally spaced points can be spectacularly bad. In this question, $f(z)$ will always refer to the function defined in (16), and $\alpha \in \mathbb{C}$ is a constant complex number.

Exercise 5.1. *Using the divided difference recurrence relation, or otherwise, prove that*

$$f[z_0, z_1, \ldots, z_m] = \frac{1}{(\alpha - z_0)(\alpha - z_1)\cdots(\alpha - z_m)}, \tag{17}$$

where $z_0, z_1, \ldots, z_m$ are different complex numbers.

This simple explicit formula enables us to analyse polynomial interpolants to $f(z)$.

Exercise 5.2. *Let $p_n \in \mathbb{P}_n$ interpolate $f(z)$ at different points $z_0, z_1, \ldots, z_n \in \mathbb{C}$. Prove that*

$$f(z) - p_n(z) = \frac{1}{\alpha - z} \prod_{k=0}^{n} \left(\frac{z - z_k}{\alpha - z_k} \right). \tag{18}$$

If we choose equally spaced points, that is, $z_k = k/n$, for $k = 0, 1, \ldots, n$, and set $z \equiv x \in \mathbb{R}$, then (5.2) becomes

$$f(x) - p_n(x) = \frac{1}{\alpha - x} \prod_{k=0}^{n} \left(\frac{x - k/n}{\alpha - k/n} \right). \tag{19}$$

Exercise 5.3. *Show that*

(20)

$$\log_e \left[|f(x) - p_n(x)|^{1/n} \right] + \frac{1}{n} \log_e |\alpha - x| = \frac{1}{n} \sum_{k=0}^{n} \log_e |x - k/n| - \frac{1}{n} \sum_{k=0}^{n} \log_e |\alpha - k/n|.$$

If we let $n \to \infty$ in (20), then we obtain

$$\lim_{n\to\infty} \log_e \left[r_n(x)^{1/n} \right] = R(x) \equiv \int_0^1 \log_e |x - t| \, dt - \int_0^1 \log_e |\alpha - t| \, dt, \tag{21}$$

where $r_n(x) = |f(x) - p_n(x)|$.

Exercise 5.4. (1) *Prove that, if $R(x) > 0$, then $r_n(x) \geq \exp(nR(x)/2)$ for all sufficiently large n.*
(2) *Prove that, if $R(x) < 0$, then $r_n(x) \leq \exp(-nR(x)/2)$ for all sufficiently large n*

The import of the last exercise is that, if we interpolate $f(z) = 1/(\alpha - z)$ at equally spaced points $\{0, 1/n, 2/n, \ldots, (n-1)/n, 1\}$, then the error $f(x) - p_n(x)$ *either* increases exponentially *or* decreases exponentially, as $n \to \infty$.

References

[1] de Boor, C. (2001), “A Practical Guide to Splines”, Springer.
[2] Conte, S. D., and C. de Boor (1980), “Elementary Numerical Analysis”, McGraw Hill.
[3] Davis, P. J. (1963), “Interpolation and Approximation”, Dover.
[4] Hairer, E. and G. Wanner (1996), “Analysis by its History”, Springer.
[5] Trefethen, L. N. (2013), “Approximation Theory and Approximation Practice”, SIAM.

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