# Newton and Polynomial Interpolation

Brad Baxter
Birkbeck College, University of London

December 14, 2023

You can download these notes from my office server

`http://econ109.econ.bbk.ac.uk/brad/teaching/Problems_in_Maths`

and my home server

`http://www.cato.tzo.com/brad_bbk/teaching/Problems_in_Maths/`

## Applied Mathematics in 1600:

Applied mathematics was expanding rapidly, from providing better tables of trigonometric functions for European oceanic navies, to improved ways to calculate interest on debts.

War and money have always been closely linked to mathematical applications!

Algebra was a Sixteenth century invention, but almost nothing was known of calculus in 1600. This was about to change!

Computation in 2020: $\$100$ phone can compute $10^8$ FLOPS (floating point operations per second).

Computation in 1990: $\$10^4$ research workstation did $10^6$ FLOPS.

Computation in the 1970s: Schoolchildren still taught to use trigonometric and logarithmic tables ($x$ is measured in degrees), e.g.

| $x$ | $\cos x$ |
|---|---|
| 0 | 1.0000 |
| 1 | 0.99985 |
| 2 | 0.99939 |
| 3 | 0.99863 |
| 4 | 0.99756 |
| 5 | 0.99619 |

Four-figure tables needed enormous work to produce in 1600.
What if we need ~~sin~~ 1.4? cos 1.4
One simple way is linear interpolation: let $f(x) = \cos x$ and define

$$p(x) = f(1) + (f(2) - f(1))\,(x - 1)\,,$$

Then

$$p(1.4) = 0.99966.$$

The true value is $\cos 1.4 = 0.9997014897811831\ldots$, so not too bad: an error of about $4 \times 10^{-5}$.
Can we do better?

## Idea: use 3 values and fit a quadratic:

Let $a = 1$, $b = 2$ and $c = 0$ and define

$$q(x) = p(x) + \underline{Q(x-a)(x-b)}.$$

Already know $p(a) = \cos a$ and $p(b) = \cos b$.
The quadratic term $(x-a)(x-b)$ vanishes at $a$ and $b$, so $q(a) = \cos a$ and $q(b) = \cos b$.
To reproduce the value at $c$, solve

$$\text{cos} \qquad \cancel{\sin} c = p(c) + Q(c-a)(c-b),$$

so

$$q(1.4) = 0.9997014954967155$$

which is *much* closer to

$$\cos 1.4 = 0.9997014897811831$$

since the error is now roughly $5.71 \times 10^{-9}$.
Quadratic interpolation reduces the error by a factor of $10^4$: excellent!

## Polynomial Interpolation

Let

$$z_0, z_1, \ldots, z_n$$

be any different complex numbers and let

$$f_0, \ldots, f_n$$

be any complex numbers (they don't need to be distinct).
Want a polynomial $p$ of degree $n$ for which

$$p(z_j) = f_j, \qquad \text{for } 0 \le j \le n.$$

Jargon: $p$ is an **interpolating polynomial**, and we say that $p$ **interpolates** the data

$$(z_0, f_0), (z_1, f_1), \ldots, (z_n, f_n).$$

Let $\mathbb{P}_n$ denote the vector space of polynomials of degree at most $n$.

Example

Find the quadratic polynomial satisfying $p(0) = \alpha$, $p(1) = \beta$ and $p(4) = \gamma$.

Could just substitute $p(x) = p_0 + p_1 x + p_2 x^2$ and solve the three linear equations to obtain the coefficients.

Simpler solution:

$$p(z) = \alpha \frac{(z-1)(z-4)}{(0-1)(0-4)} + \beta \frac{z(z-4)}{(1-0)(1-4)} + \gamma \frac{z(z-1)}{(4-0)(4-1)}.$$

Key point:

$$\ell_0(z) = \frac{(z-1)(z-4)}{(0-1)(0-4)} = \frac{1}{4}(z-1)(z-4)$$

satisfies

$$\ell_0(1) = \ell_0(4) = 0 \quad \text{and} \quad \ell_0(0) = 1.$$

If $z_0 = 0$, $z_1 = 1$ and $z_2 = 4$, then

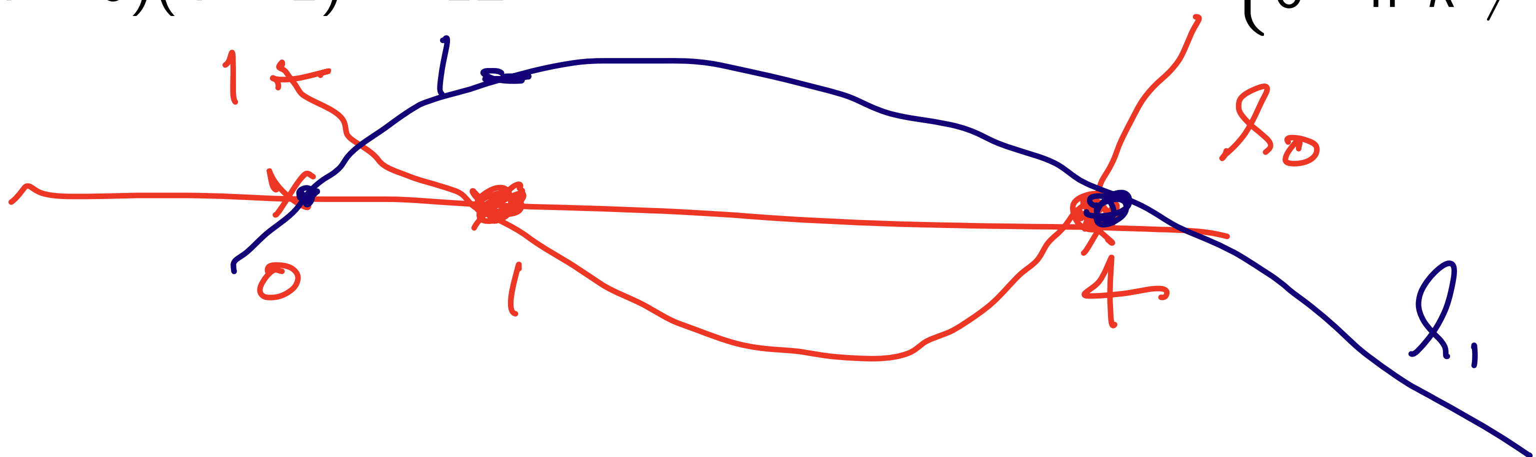
$$\ell_0(z_k) = \begin{cases} 1 & \text{if } k = 0, \\ 0 & \text{if } k \neq 0. \end{cases}$$

Thus

$$\ell_1(z) = \frac{z(z-4)}{(1-0)(1-4)} = -\frac{1}{3}z(z-4) \quad \text{and} \quad \ell_1(z_k) = \begin{cases} 1 & \text{if } k = 1, \\ 0 & \text{if } k \neq 1, \end{cases}$$

while

$$\ell_2(z) = \frac{z(z-1)}{(4-0)(4-1)} = \frac{1}{12}z(z-1) \quad \text{and} \quad \ell_2(z_k) = \begin{cases} 1 & \text{if } k = 2, \\ 0 & \text{if } k \neq 2. \end{cases}$$



## Lemma

$$\ell_j(z) = \frac{(z - z_0)(z - z_1)\cdots(z - z_{j-1})(z - z_{j+1})\cdots(z - z_n)}{(z_j - z_0)(z_j - z_1)\cdots(z_j - z_{j-1})(z_j - z_{j+1})\cdots(z_j - z_n)},$$

or, more briefly,

$$\ell_j(z) = \prod_{k=0, k\neq j}^{n} \frac{z - z_k}{z_j - z_k}, \qquad 0 \le j \le n.$$

Then

$$\ell_j(z_i) = \begin{cases} 1 & \text{if } i = j, \\ 0 & \text{if } i \neq j. \end{cases}$$

## Lagrange form of interpolating polynomial

The interpolating polynomial $p \in \mathbb{P}_n$ for the data $\{(z_j, f_j) : 0 \le j \le n\}$ is

$$p(z) = \sum_{j=0}^{n} f_j \ell_j(z), \qquad z \in \mathbb{C}.$$

## Proof:

$$p(z_k) = \sum_{j=0}^{n} f_j \ell_j(z_k) = f_k, \qquad 0 \le k \le n.$$

There is exactly one interpolating polynomial $p \in \mathbb{P}_n$ when the points $z_0, z_1, \ldots, z_n$ are distinct.

Proof:

Existence: the Lagrange form of the interpolating polynomial.

Uniqueness: let $p$ and $q$ be interpolating polynomials of degree $n$. Their difference $p - q$ is a polynomial of degree $n$ that vanishes at the $n + 1$ different points $z_0, \ldots, z_n$. Hence $p - q$ vanishes identically.

Linear interpolation is easy The linear polynomial interpolating $f$ at $z_0$ and $z_1$ is

$$p_1(z) = f(z_0) + f[z_0, z_1](z - z_0),$$

where

$$f[z_0, z_1] = \frac{f(z_1) - f(z_0)}{z_1 - z_0}.$$

Exercise: Check this!

$f[z_0, z_1]$ is our first divided difference.

## Newton's brilliant laziness

Suppose we have computed $p_{n-1} \in \mathbb{P}_{n-1}$ interpolating data $\{(z_j, f_j) : 0 \le j \le n-1\}$. We then obtain new data: $(z_n, f_n)$.

Key idea:

$$p_n(z) = p_{n-1}(z) + \overbrace{C(z - z_0)(z - z_1)\cdots(z - z_{n-1})}^{=0 \text{ for } 0 \le j \le n-1},$$

Then

$$p_n(z_j) = p_{n-1}(z_j) = f_j, \qquad \text{for } 0 \le j \le n-1.$$

Choose $C$ using

$$f_n = p_{n-1}(z_n) + C \prod_{k=0}^{n-1} (z_n - z_k).$$

Obviously $C$ depends on $f$ and $z_0, z_1, \ldots, z_n$. Newton's notation was

$$C = f[z_0, z_1, \ldots, z_n].$$

So

$$p_n(z) = p_{n-1}(z) + f[z_0, z_1, \ldots, z_n](z - z_0)(z - z_1)\cdots(z - z_{n-1}).$$

Jargon: $f[z_0, \ldots, z_n]$ is a **divided difference**.

Proposition:

$$f[z_0, z_1, \ldots, z_n] = \sum_{j=0}^{n} \frac{f(z_j)}{\prod_{k=0,k\neq j}^{n}(z_j - z_k)}$$

and $f[z_0, \ldots, z_n] = 0$ when $f$ is a polynomial of degree less than $n$.

Proof:

Look at the coefficients of $z^n$ in the Lagrange form

$$p_n(z) = \sum_{j=0}^{n} f(z_j)\ell_j(z)$$

and the Newton form.

If $f(z) = z^\ell$ and $\ell < n$, then the coefficient of $z^n$ in $p_n$ is zero, but the coefficient of $z^n$ in the Newton form is $f[z_0, \ldots, z_n]$.

Newton form summary:

$$
\begin{aligned}
p_n(z) = \overbrace{f[z_0]}^{f(z_0)} + \overbrace{f[z_0, z_1]}^{\frac{f(z_1) - f(z_0)}{z_1 - z_0}}(z - z_0) + f[z_0, z_1, z_2](z - z_0)(z - z_1) + \cdots \\
+ f[z_0, z_1, \ldots, z_n](z - z_0)(z - z_1) \cdots (z - z_{n-1}).
\end{aligned}
$$

Example:

The Newton form of the quadratic polynomial satisfying $p(0) = f(0)$, $p(1) = f(1)$ and $p(4) = f(4)$ is

$$p(z) = f[0] + f[0,1]z + f[0,1,4]z(z-1).$$

You'll see how to calculate the coefficients shortly.

## Divided Difference Recurrence Relation:

For any distinct complex numbers $z_0, z_1, \ldots, z_n, z_{n+1}$

$$f[z_0, \ldots, z_{n+1}] = \frac{f[z_0, \ldots, z_n] - f[z_1, \ldots, z_{n+1}]}{z_0 - z_{n+1}}.$$

## Proof:

Introduce two polynomials: (i) $p \in \mathbb{P}_n$ interpolates $\{(z_k, f_k) : 0 \le k \le n\}$, and (ii) $q \in \mathbb{P}_n$ interpolates $\{(z_k, f_k) : 1 \le k \le n+1\}$.

The coefficients of highest degree for $p$ and $q$ are $f[z_0, \ldots, z_n]$ and $f[z_1, \ldots, z_{n+1}]$, respectively.

Trick: Define

$$r(z) = \frac{(z - z_{n+1})p(z) - (z - z_0)q(z)}{z_0 - z_{n+1}}.$$

Exercise: Check that $r(z)$ is a polynomial of degree $n$ and interpolates $f$ at $z_0, \ldots, z_{n+1}$

The coefficient of highest degree in $r$ is $f[z_0, \ldots, z_{n+1}]$, so equate the coefficients of highest degree to obtain the divided difference relation.

# The divided difference table

$$
\begin{tikzcd}
f(x_0) \arrow[dr] & & \\
& f[x_0, x_1] \arrow[dr] & \\
f(x_1) \arrow[ur] \arrow[dr] & & f[x_0, x_1, x_2] \\
& f[x_1, x_2] \arrow[ur] & \\
f(x_2) \arrow[ur] & &
\end{tikzcd}
$$

$$f[x_0, x_1] = \frac{f(x_1) - f(x_0)}{x_1 - x_0}$$

$$f[x_0, x_1, x_2] = \frac{f[x_1, x_2] - f[x_0, x_1]}{x_2 - x_0}$$

$$f[x_1, x_2] = \frac{f(x_2) - f(x_1)}{x_2 - x_1}$$

Example: Let $f(x) = x^3$ and let $x_0 = 0$, $x_1 = 1$ and $x_2 = 4$.

| $x_k$ | $f(x) = x^3$ | | |
|---|---|---|---|
| $0$ | $0$ | | |
| | | $\frac{1-0}{1-0} = 1$ | |
| $1$ | $1$ | | $\frac{21-1}{4-0} = 5$ |
| | | $\frac{64-1}{4-1} = 21 = f[1,4]$ | |
| $4$ | $64$ | | |

$$f[x_0, x_1, x_2] = \frac{f[x_1, x_2] - f[x_0, x_1]}{x_2 - x_0}$$

$\parallel$

Thus $f[x_0, x_1] = 1$, $f[x_0, x_1, x_2] = 5$ and the Newton form of the quadratic interpolating $f(x) = x^3$ at 0, 1 and 4 is by

$$p(x) = x + 5x(x - 1).$$

$p(4) = 4 + 5 * 4 * 3 = 64$

## Error in polynomial interpolation

Let $p \in \mathbb{P}_n$ interpolate $f$ at $n$ distinct complex numbers $z_0, z_1, \ldots, z_n$. The error $e = f - p$ satisfies

$$e(w) = f[z_0, z_1, \ldots, z_n, w] \prod_{k=0}^{n} (w - z_k), \qquad w \in \mathbb{C}.$$

Proof: Add a new interpolation point $z_{n+1}$. The new Newton interpolating polynomial $q \in \mathbb{P}_{n+1}$ is

$$q(z) = p(z) + f[z_0, z_1, \ldots, z_n, z_{n+1}] \prod_{k=0}^{n} (z - z_k).$$

Hence

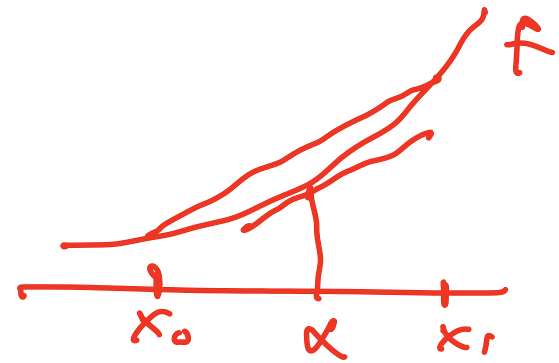
$$f(z_{n+1}) = p(z_{n+1}) + f[z_0, z_1, \ldots, z_n, z_{n+1}] \prod_{k=0}^{n} (z_{n+1} - z_k).$$

Replace $z_{n+1}$ by $w$.

$p_1(x) = f(x_0) + f[x_0, x_1](x - x_0)$

let $e_1(x) = f(x) - p_1(x)$

## The first mean value theorem:




$$f[x_0, x_1] = \frac{f(x_1) - f(x_0)}{x_1 - x_0} = f'(\alpha),$$

for some point $\alpha \in [x_0, x_1]$.

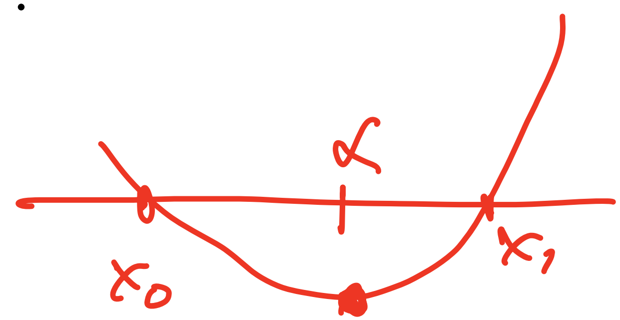
## Divided difference MVT

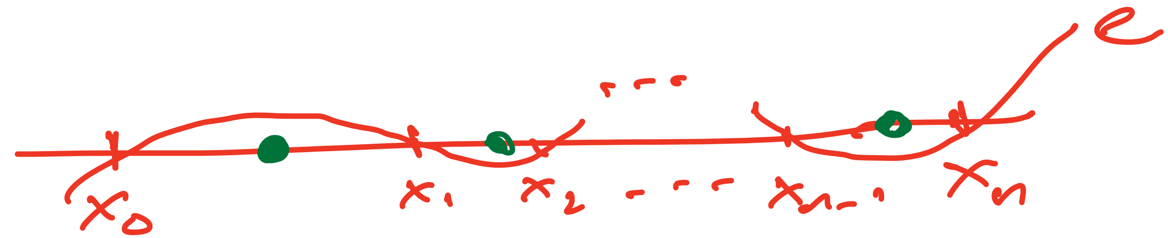
Let $f$ have continuous $(n+1)$st derivative and let $x_0 < x_1 < \cdots < x_n$ be **real** numbers. Then there is a point $\alpha \in [x_0, x_n]$ such that

$e_1'(x) = f'(x) - f[x_0, x_1]$

$$f[x_0, x_1, \ldots, x_n] = \frac{f^n(\alpha)}{n!}.$$

$e_1(x_0) = e_1(x_1) = 0$






Proof:

Let $p_n \in \mathbb{P}_n$ interpolate $f$ at $x_0, \ldots, x_n$. Then the error function $e = f - p_n$ has at least $n+1$ zeros in $[x_0, x_n]$. Hence its derivative $e'$ has at least $n$ zeros in $[x_0, x_n]$, and its second derivative $e''$ has at least $n-1$ zeros, ...

Hence $e^{(n)}$ has at last one zero, $\alpha$ say, in $[x_0, x_n]$.

But then

$$0 = e^{(n)}(\alpha) = f^{(n)}(\alpha) - f[x_0, \ldots, x_n]n!.$$

Let $f$ have continuous $(n+1)$st derivative and let $x_0, x_1, \ldots, x_n$ be different real numbers. If $p_n \in \mathbb{P}_n$ is the interpolating polynomial, then the error $e_n = f - p_n$ satisfies

$$|e_n(x)| \le \frac{M \prod_{k=0}^{n} |x - x_k|}{(n+1)!}, \qquad x \in [a, b],$$

where

$$M = \max\{|f^{(n+1)}(t)| : a \le t \le b\}.$$

Let $f(x) = \exp(x)$ and let $a = -1/2$, $b = 1/2$. If the interpolation points are all in the interval $[-1/2, 1/2]$, then the error of interpolation satisfies

$$|e_n(x)| \le \frac{e}{(n+1)!}, \qquad -1/2 \le x \le 1/2.$$

In other words, the error is tiny.