1.

(a) We have $p(\alpha) = a_0 + a_1\alpha = f(\alpha)$ (1)

and $p(-\alpha) = a_0 - a_1\alpha = f(-\alpha)$. (2)

Hence $a_0 = \frac{1}{2}(f(\alpha) + f(-\alpha))$ (3)

and $a_1 = \dfrac{f(\alpha) - f(-\alpha)}{2\alpha}$. (4)

Then

$$\int_{-h}^{h} a_0 + a_1 x \, dx = a_0 \int_{-h}^{h} dx$$

$$= 2h a_0$$

$$= h(f(\alpha) + f(-\alpha)).$$

(b) $\int_{-h}^{h} x^k \, dx = 0$ for any odd $k \in \mathbb{Z}_+$

and $h(\alpha^k + (-1)^k \alpha^k) = 0$,

Hence we only need to find $\alpha$ s.t.

$$\int_{-h}^{h} x^2 \, dx = 2h\alpha^2$$

i.e. $\dfrac{2h^3}{3} = 2h\alpha^2$

or $\alpha = \dfrac{\pm h}{\sqrt{3}}$.

Thus
$$\int_{-h}^{h} f(x)\,dx = h\left(f\left(\frac{h}{\sqrt{3}}\right) + f\left(\frac{-h}{\sqrt{3}}\right)\right)$$
is exact for any $f \in \mathbb{P}_3$.

BUT if $f(x) = x^4$, then
$$\int_{-h}^{h} x^4\,dx = 2\int_0^h x^4\,dx = \frac{2h^5}{5}$$
while
$$h\left[\left(\frac{h}{\sqrt{3}}\right)^4 + \left(\frac{-h}{\sqrt{3}}\right)^4\right] = \frac{2h^5}{9}$$
so it's not exact for quartics.

2(a) The geometric series $1+z+z^2+\cdots+z^{n-1} = \dfrac{z^n-1}{z-1}$

implies
$$\ell_0(z) = \frac{1}{n}\left(\frac{z^n-1}{z-1}\right) = \frac{1}{n}(1+z+\cdots+z^{n-1}).$$

Thus $\ell_0 \in \mathbb{P}_{n-1}$ and
$$\ell_0(1) = \frac{1}{n}(1+1+\cdots+1) = 1.$$

Alternatively, by de l'Hôpital's rule,
$$\lim_{z\to 1} \ell_0(z) = \frac{1}{n} \lim_{z\to 1} n z^{n-1} = 1.$$

If $z = \omega^j$, $1 \le j \le n-1$, then $z-1 \ne 0$

and $(\omega^j)^n - 1 = 0$. Hence $\ell_0(\omega^k) = 0$
for $1 \le k \le n-1$.

(b)
$$\ell_0(z) = \frac{1}{n}\left(\frac{z^n-1}{z-1}\right)$$
$$= \frac{1}{n}\,\frac{(z-1)(z-\omega)\cdots(z-\omega^{n-1})}{z-1}$$
$$= \frac{(z-\omega)(z-\omega^2)\cdots(z-\omega^{n-1})}{n}.$$

BUT
$$\lim_{z\to 1} \prod_{k=1}^{n-1}(z-\omega^k) = \lim_{z\to 1}\frac{z^n-1}{z-1} = n,$$

i.e.
$$\prod_{k=1}^{n-1}(1-\omega^k) = n.$$

Hence
$$\ell_0(z) = \prod_{k=1}^{n-1}\left(\frac{z-\omega^k}{1-\omega^k}\right).$$

OR : Note that $l_0$ is the Lagrange interpolating poly, hence unique.

(c) $l_j(\omega^k) = l_0(\omega^{k-j}) = \begin{cases} 1 & k=j \\ 0 & \text{else.} \end{cases}$

(d) $p(z) = \sum_{j=0}^{n-1} \omega^{-j} l_j(z)$

satisfies $p(z) = \frac{1}{z}$ when $z = \omega^j$.

Now $z^n = 1 \Rightarrow z^{n-1} = \frac{1}{z}$ for $z = \omega^j$,

so $p(z) \equiv z^{n-1}$ by uniqueness of the interpolating polynomial

3.

(a)

| $x$ | $f(x)=x^3$ | 1st | 2nd | 3rd | 4th |
| --- | --- | --- | --- | --- | --- |
| 0 | 0 | | | | |
| | | 1 | | | |
| | | | 3 | | |
| 1 | 1 | | | 1 | |
| | | 7 | | | 0 $= f[0,1,2,3,4]$ |
| | | | 6 | | |
| 2 | 8 | | | 1 | |
| | | 19 | | | |
| | | | 9 | | |
| 3 | 27 | | | | |
| | | $64-27$ $=37$ | | | |
| 4 | 64 | | | | |

(b)
$$\begin{aligned}\Delta^2 f(x) &= \Delta(\Delta f(x)) \\ &= \Delta f(x+1) - \Delta f(x) \\ &= f(x+2) - f(x+1) - (f(x+1) - f(x)) \\ &= f(x+2) - 2f(x+1) + f(x).\end{aligned}$$

(c)
$$\begin{aligned}\Delta^3 f(x) &= (E-1)^3 f(x) \\ &= (E^3 - 3E^2 + 3E - E)\, f(x)\end{aligned}$$
(Binomial Theorem)
$$= f(x+3) - 3f(x+2) + 3f(x+1) - f(x)$$

(d) The result is true for $n=1$, 2 (3(b)) and 3 (3(c)). If it's true for $n \geq 1$, then

$f[x, x+1, \ldots, x+n, x+n+1]$

$$= \frac{f[x+1, \ldots, x+n+1] - f[x, x+1, \ldots, x+n]}{\cancel{x}+n+1 - \cancel{x}}$$

$$= \frac{f[x+1, (x+1)+1, \ldots, (x+1)+n] - f[x, \ldots, x+n]}{n+1}$$

$$= \frac{\dfrac{\Delta^n f(x+1)}{n!} - \dfrac{\Delta^n f(x)}{n!}}{n+1} \quad \text{(induction hypothesis)}$$

$$= \frac{\Delta^{n+1} f(x)}{(n+1)!}$$

Hence the result is true by induction.