2025-1-4

Paper 1, Section I

4C Variational Principles

Given a real symmetric $n \times n$ matrix A, consider the quadratic function

$$Q(\mathbf{x}) = \mathbf{x}^T A \mathbf{x}\,,$$

on the unit sphere $S^{n-1} = \{\mathbf{x} \in \mathbb{R}^n : \mathbf{x}^T\mathbf{x} = 1\}$. Assume that $\mathbf{x_0}$ is a unit vector such that

$$Q(\mathbf{x_0}) \geqslant Q(\mathbf{x})\,, \quad \forall\, \mathbf{x} \in S^{n-1}\,.$$

Show that $\mathbf{x_0}$ is an eigenvector of the matrix A and determine the corresponding eigenvalue E. How does this eigenvalue compare to the other eigenvalues of A?

For the case that

$$A = \begin{pmatrix} 1 & t \\ t & 1 \end{pmatrix} \qquad -\infty < t < +\infty$$

calculate E, as a function of $t \in \mathbb{R}$, and draw a sketch to show that it is convex.

We solve:

$$\min \quad Q(\underline{x}) = \underline{x}^T A \underline{x}$$

$$\text{s.t.} \quad g(\underline{x}) = \underline{x}^T \underline{x} = 1.$$

Then

$$\underline{\nabla} Q(\underline{x}_0) = \lambda \underline{\nabla} g(\underline{x}_0)$$

is the equation

$$2A\underline{x}_0 = 2\lambda \underline{x}_0,$$

i.e.

$$A\underline{x}_0 = \lambda \underline{x}_0.$$

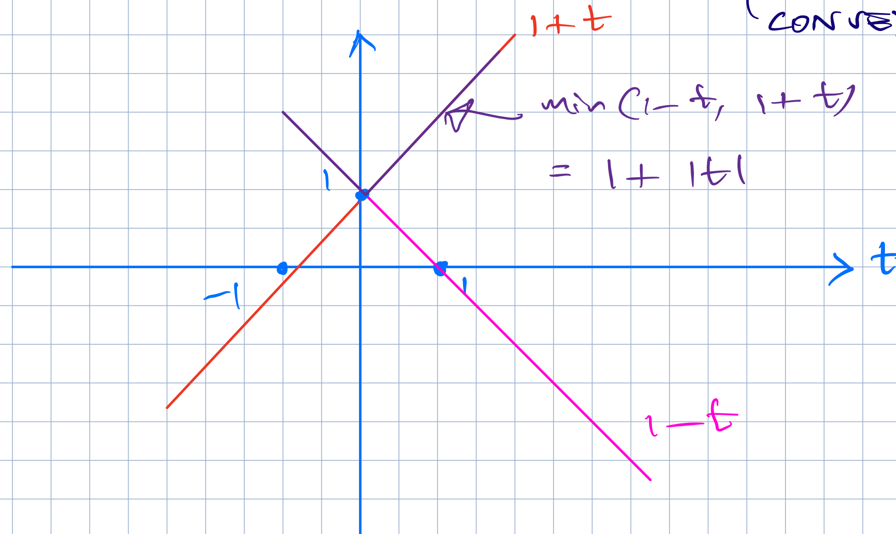
For any other unit eigenvector $\underline{v}$, i.e. $A\underline{v} = \mu \underline{v}$, we must have $Q(\underline{v}) = \underline{v}^T A \underline{v} = \mu \geqslant \lambda$ ~~&~~ $\lambda = \underline{x}_0^T A \underline{x}_0$. This question was set by a physicist, so they have written $\lambda \equiv E$.

$A = \begin{pmatrix} 1 & t \\ t & 1 \end{pmatrix}$. Then $A\begin{pmatrix} 1 \\ 1 \end{pmatrix} = (1+t)\begin{pmatrix} 1 \\ 1 \end{pmatrix}$

& $A\begin{pmatrix} -1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 & t \\ t & 1 \end{pmatrix}\begin{pmatrix} -1 \\ 1 \end{pmatrix} = \begin{pmatrix} -1+t \\ -t+1 \end{pmatrix} = (1-t)\begin{pmatrix} -1 \\ 1 \end{pmatrix}$.

Thus $E = \min(1-t,\ 1+t) = 1+|t|$.

$\uparrow$ CONVEX



OR: $A = (1-t)I + t\underline{e}\underline{e}^T$, $\underline{e} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$,

so $A\underline{e} = (1-t)\underline{e} + 2t\underline{e} = (1+t)\underline{e}$

& $\underline{v}^T\underline{e} = 0 \Rightarrow A\underline{v} = (1-t)\underline{v}$.

CLAIM: $f: \mathbb{R} \longrightarrow \mathbb{R}$, $f(x) = |x|$, $x \in \mathbb{R}$, is convex.

Proof: We must show that, for any $x, y \in \mathbb{R}$, we have

$$f(tx + (1-t)y) \leq t f(x) + (1-t) f(y)$$

for $0 \leq t \leq 1$. Now $RHS = t|x| + (1-t)|y|$

$$LHS = |tx + (1-t)y|$$

$$\leq t|x| + (1-t)|y| = RHS. \quad \square$$