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Paper 2, Section II

13C Variational Principles

This question concerns the movement of a particle in space $\mathbb{R}^3$. Introduce cylindrical coordinates (ρ, ϕ, z) and assume that the trajectory of the particle can be parameterized as a curve

$$z \mapsto (\rho(z), \phi(z))$$

going from $A = (\rho(z_0), \phi(z_0), z_0)$ to $B = (\rho(z_1), \phi(z_1), z_1)$, and is such as to make the following functional stationary:

$$F[\rho, \phi] = \int_{z_0}^{z_1} n(\rho, \phi, z) \sqrt{1 + \left(\frac{\partial \rho}{\partial z}\right)^2 + \rho^2 \left(\frac{\partial \phi}{\partial z}\right)^2}\, dz\,, \quad \text{where } z_1 > z_0\,,$$

where the function $n = n(\rho, \phi, z)$ is positive and smooth. Write down the Euler-Lagrange equations for this functional.

In the case that $n = n(\rho)$ depends only on ρ, show that there are special solutions to the Euler-Lagrange equations of the form

$$\rho(z) = R, \qquad \phi(z) = \phi_0 + \omega(R) z,$$

where R and ϕ_0 are constants, and $\omega = \omega(R)$ solves an equation

$$n(R) R \omega^2 + a(1 + R^2\omega^2) n'(R) = 0 \qquad (*)$$

for some constant a which you should find. [*You may assume $(*)$ has two solutions $\pm\omega$, with $\omega > 0$.*]

Find a condition on a positive number L which implies that the points having cylindrical coordinates $(R, \phi_0, 0)$ and (R, ϕ_0, L) can be joined by means of these special solutions and sketch two of them.

$$F(\rho, \phi) = \int_{z_0}^{z_1} n\left(1 + \rho_z^2 + \rho^2 \phi_z^2\right)^{\frac{1}{2}} dz$$

E-L:

$$\frac{\partial F}{\partial \rho} - \frac{d}{dz}\left(\frac{\partial F}{\partial \rho_z}\right) = 0\,, \qquad (1)$$

$$\frac{\partial F}{\partial \phi} - \frac{d}{dz}\left(\frac{\partial F}{\partial \phi_z}\right) = 0\,. \qquad (2)$$

(1):

$$0 = n\left(1 + \rho_z^2 + \rho^2\phi_z^2\right)^{-\frac{1}{2}} \phi_z^2 \rho - \frac{d}{dz}\left[n\left(1 + \rho_z^2 + \rho^2\phi_z^2\right)^{-\frac{1}{2}} \rho_z\right]$$
$$+ n_\rho \left(1 + \rho_z^2 + \rho^2\phi_z^2\right)^{\frac{1}{2}} \qquad (3)$$

(2):

$$0 = n_\phi (1+\rho_z^2+\rho^2\phi_z^2)^{\frac{1}{2}} - \frac{d}{dz}\left[n(1+\rho_z^2+\rho^2\phi_z^2)^{-\frac{1}{2}}\rho^2\phi_z\right]. \quad (4)$$

If $n \equiv n(\rho)$, $\rho(z) \equiv R$, then (3) becomes

$$0 = n(1+R^2\phi_z^2)^{-1/2}\phi_z^2 R + n'(1+R^2\phi_z^2)^{\frac{1}{2}}$$

& (4) becomes

$$C = n(\rho)(1+R^2\phi_z^2)^{-\frac{1}{2}} R^2 \phi_z . \quad (5)$$

If $\phi(z) = \phi_0 + w(R)z$, then $\phi_z = w(R)$ &

$$C = n(R)(1+R^2w^2)^{-\frac{1}{2}} R^2 w$$ ← RHS obviously constant already.

while (3) $\mapsto$

$$0 = n(R)(1+R^2w^2)^{-1/2} w^2 R + n'(R)(1+R^2w^2)^{\frac{1}{2}}$$

$$\rightarrow 0 = n(R) w^2 R + n'(R)(1+R^2w^2). \quad (*) \quad (6)$$

SO $a = 1$!

Then

$$w^2[n(R)R + n'(R)R^2] = -n'(R),$$

& we are told there are 2 roots $\pm w$, $w>0$.

We now have $\phi(z) = \phi_0 \pm wz$

so need to solve

$$\phi(L) = \phi_0 \pm wL = \phi_0 + 2k\pi, \; k\in\mathbb{Z},$$

i.e. $\pm wL = 2k\pi$ for some $k \in \mathbb{Z}$.

or $L = \pm \dfrac{2k\pi}{w}$, $k\in\mathbb{Z}$

All of these are integer multiples of $\frac{2\pi}{w}$, so have motion on a helix either UP $(+\omega)$ or DOWN $(-\omega)$