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Paper 3, Section I

4C Variational Principles

Consider the functional

$$S[x,y] = \frac{1}{2}\int_{t_0}^{t_1}\left(\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + 2\sin\omega t\,\frac{dx}{dt} - y^2\right)dt$$

defined on smooth curves $t \mapsto (x(t), y(t))$ in the plane. Assume $\omega \in \mathbb{R}$ is constant.

Write out the Euler-Lagrange equations for S and find the general solution.

What symmetries does the system have?

Find all the first integrals (conserved quantities) of the system.

[*You may either use the Noether theorem or work with the Euler-Lagrange equations. Consider all values of* $\omega \in \mathbb{R}$.]

$$f(t, x, y, \dot{x}, \dot{y}) = \frac{1}{2}(\dot{x}^2 + \dot{y}^2 + 2\sin\omega t\,\dot{x} - y^2).$$

Euler-Lagrange equations are

$$0 = \frac{\partial f}{\partial x} - \frac{d}{dt}\left(\frac{\partial f}{\partial \dot{x}}\right) \quad (1)$$

& $$0 = \frac{\partial f}{\partial y} - \frac{d}{dt}\left(\frac{\partial f}{\partial \dot{y}}\right), \quad (2)$$

Then (1) $\Rightarrow$

$$0 = -\frac{d}{dt}(\dot{x} + 2\sin\omega t)$$

or $$0 = \ddot{x} - 2\omega\cos\omega t \quad (3)$$

& $$0 = -y - \frac{d}{dt}(\dot{y})$$

or $$0 = \ddot{y} + y. \quad (4)$$

Now (4) $\Rightarrow$ $y(t) = P\cos t + Q\sin t$ (5)

while (3) $\Rightarrow$

$$\dot{x} = \sin \omega t + A$$

$$\Rightarrow \quad x(t) = \frac{-\cos \omega t}{\omega} + At + B. \quad (6)$$

Conserved quantities:

1. $\omega = 0 \iff \frac{\partial f}{\partial t} = 0 \Rightarrow 1^{st}$ integral conserved.

2. Conjugate momentum is conserved:

$$p_x \equiv \frac{\partial L}{\partial \dot{x}} = \dot{x} + \sin \omega t$$