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Paper 4, Section II

13C Variational Principles

The equation of motion for a bead of mass m moving without friction on a cycloidal shaped wire is the Euler-Lagrange equation for the functional

$$S[\phi] = \int_0^T \Big(ma^2(1-\cos\phi)\dot\phi^2 - mga(1+\cos\phi)\Big)\, dt\,, \qquad T>0\,.$$

Write down the Euler-Lagrange equation for this functional, and show it implies that $u=\cos(\frac{\phi}{2})$ satisfies

$$\ddot u + \omega^2 u = 0, \qquad (*)$$

where ω^2 is a positive number which you should find. [*You should take m, g, a to be positive constants.*]

Using the change of dependent variable $\phi \to u = \cos(\frac{\phi}{2})$, define a new functional $\hat S[u] = \int_0^T f(u,\dot u)\, dt$ such that $\hat S[u] = S[\phi]$; give a formula for f and give the Euler-Lagrange equation for $\hat S$. How is this equation related to $(*)$?

Give the second variation functional $\delta^2 \hat S(\eta)$, where the variation functions η vanish at the endpoints $t=0$ and $t=T$. Consider the solution $u(t) = A\cos\omega t$ of $(*)$ with fixed endpoint conditions

$$u(0) = A, \qquad u(T) = A\cos\omega T,$$

on the interval $0 \leqslant t \leqslant T$. By considering the orthonormal collection of functions

$$e_n(t) = \sqrt{\frac{2}{T}}\sin\frac{n\pi t}{T}\,,$$

find a number t_0 such that $A\cos\omega t$ is a local minimizer of $\hat S$ if $T<t_0$ but not for $T>t_0$.

[*Hint: you may assume all variations to be of the form $\eta = \sum_{n=1}^\infty c_n e_n(t)$, and rearrange and interchange sums with derivatives as needed. Observe that $\ddot e_n = -(n\pi/T)^2 e_n$.*]

$F = ma^2(1-\cos\phi)\dot\phi^2 - mga(1+\cos\phi)$

E-L:

$$0 = \frac{\partial F}{\partial \phi} - \frac{d}{dt}\left(\frac{\partial F}{\partial \dot\phi}\right)$$

$$= ma^2 \sin\phi \cdot \dot\phi^2 + mga\sin\phi - \frac{d}{dt}\left(2ma^2(1-\cos\phi)\dot\phi\right).$$

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Then

$$0 = \sin\phi.\left(\dot\phi^2 + \frac{g}{a}\right) - \frac{d}{dt}\left(2(1-\cos\phi).\dot\phi\right). \qquad (2)$$

So $u = \cos\frac{\phi}{2} \Rightarrow \dot u = -\frac{1}{2}\sin\frac{\phi}{2}.\dot\phi$

$$\Rightarrow \dot\phi = \frac{-2\dot u}{\sin\phi/2}.$$

Further, $\ddot u = -\frac{1}{4}\cos\frac{\phi}{2}.\dot\phi^2 - \frac{1}{2}\sin\frac{\phi}{2}.\ddot\phi$. $\qquad (3)$

Now (2) $\rightarrow$

$$0 = \sin\phi\left(\dot\phi^2 + \frac{g}{a}\right) - 2\left(\sin\phi.\dot\phi^2 + (1-\cos\phi)\ddot\phi\right)$$

$$= \underbrace{\frac{g}{a}\sin\phi - \sin\phi.\dot\phi^2}_{2\sin\frac{\phi}{2}\cos\frac{\phi}{2}\left(\frac{g}{a}-\dot\phi^2\right)} - 2\underbrace{(1-\cos\phi)}_{2\sin^2\frac{\phi}{2}}\ddot\phi$$

$\Rightarrow$

$$0 = \cos\frac{\phi}{2}\left(\frac{g}{a}-\dot\phi^2\right) - 2\sin\frac{\phi}{2}.\ddot\phi$$

$$= \frac{g}{a}.u + 4\underbrace{\left(-\frac{1}{4}\cos\frac{\phi}{2}.\dot\phi^2 - \frac{1}{2}\sin\frac{\phi}{2}.\ddot\phi\right)}_{\ddot u} \quad \text{(using (3))}$$

OK $\qquad \ddot u + \left(\frac{g}{4a}\right)u = 0$, i.e. $\omega^2 = \frac{g}{4a}$.

This has general solution

$$u(t) = P\cos\omega t + Q\sin\omega t$$
$$= \alpha\cos(\omega t - \beta),$$

so $u = \cos\frac{\phi}{2} = \alpha\cos(\omega t - \beta)$,

i.e. $\alpha = 1$ and $\phi = 2\omega t - \gamma$, some const. γ.

Now $\quad u = \cos\frac{\phi}{2} \quad \Rightarrow \quad \dot{u} = -\frac{1}{2}\sin\frac{\phi}{2}\,.\,\dot{\phi}$

&

$$\begin{aligned}
f &= ma^2(1-\cos\phi)\,.\,\dot{\phi}^2 - mga(1+\cos\phi)\\
&= ma^2\left\{(1-\cos\phi)\dot{\phi}^2 - \left(\frac{g}{a}\right)(1+\cos\phi)\right\}\\
&= ma^2\left\{2\sin^2\frac{\phi}{2}\,.\,\dot{\phi}^2 - \left(\frac{g}{a}\right).\,2\cos^2\frac{\phi}{2}\right\}\\
&= ma^2\left\{2\sin^2\frac{\phi}{2}\,.\,\frac{4\dot{u}^2}{\sin^2\frac{\phi}{2}} - \left(\frac{g}{a}\right).\,2\cos^2\frac{\phi}{2}\right\}\\
&= ma^2\left\{8\dot{u}^2 - 2\left(\frac{g}{a}\right)u^2\right\}\\
&= 8ma^2\left\{\dot{u}^2 - \left(\frac{g}{4a}\right)u^2\right\}\\
&= 8ma^2\left\{\dot{u}^2 - \omega^2u^2\right\} \equiv f(u,\dot{u}).
\end{aligned}$$

So $\quad \hat{S}(u) = 8ma^2\int_0^T \dot{u}^2 - \omega^2u^2 \; dt.$

The new E-L is

$$\begin{aligned}
0 &= \frac{\partial f}{\partial u} - \frac{d}{dt}\left(\frac{\partial f}{\partial \dot{u}}\right)\\
&= -2\omega^2u - \frac{d}{dt}(2\dot{u})\\
&= -2\omega^2u - 2\ddot{u}
\end{aligned}$$

$\Rightarrow \quad 0 = \ddot{u} + \omega^2 u, \quad$ i.e. $(*)$ again.

2^{nd} variation :

$$\delta^2 f(\varepsilon) = \int_0^T \begin{pmatrix} \varepsilon \\ \dot\varepsilon \end{pmatrix}^T \underbrace{\begin{pmatrix} f_{uu} & f_{u\dot u} \\ f_{u\dot u} & f_{\dot u \dot u} \end{pmatrix}}_{H} \begin{pmatrix} \varepsilon \\ \dot\varepsilon \end{pmatrix} dt$$

&

$$H = \begin{pmatrix} -2\omega^2 & 0 \\ 0 & 2 \end{pmatrix}$$

i.e.

$$\delta^2 f(\varepsilon) = \int_0^T -2\omega^2 \varepsilon^2 + 2\dot\varepsilon^2 \, dt$$

with $\varepsilon(0) = 0 = \varepsilon(T)$

Then

$$\varepsilon(t) = \sum_{n=1}^{\infty} \varepsilon_n \sin \frac{n\pi t}{T}$$

&

$$\delta^2 f(\varepsilon) = -2\omega^2 \sum_{n=1}^{\infty} \varepsilon_n^2 + 2 \sum_{n=1}^{\infty} \varepsilon_n^2 \cdot \frac{n^2\pi^2}{T^2}$$

$$= 2 \sum_{n=1}^{\infty} \varepsilon_n^2 \left(\frac{n^2\pi^2}{T^2} - \omega^2 \right).$$

It's a local minimizer if $\frac{n^2\pi^2}{T^2} - \omega^2 > 0 \quad \forall n \in \mathbb{Z}_+$

i.e., $\frac{\pi^2}{T^2} > \omega^2$

or $T^2 < \frac{\pi^2}{\omega^2}$

i.e $T < \frac{\pi}{\omega}$.

Hence $T < \frac{\pi}{\omega} \Rightarrow$ local min, but NOT for $T > \frac{\pi}{\omega}$.