

1. $\phi(x_1, x_2, x_3) = \frac{1}{4}(x_1^4 + x_2^4 + x_3^4) - x_2x_3 - x_3x_1 - x_1x_2.$

So

$$\nabla \phi(\underline{x}) = \begin{pmatrix} x_1^3 - x_2 - x_3 \\ x_2^3 - x_3 - x_1 \\ x_3^3 - x_1 - x_2 \end{pmatrix}$$

Then $\nabla \phi(\underline{x}) = 0 \Rightarrow$

$$x_1^3 - x_2 = x_2 - x_1$$

i.e. $(x_1 - x_2)(x_1^2 + x_1x_2 + x_2^2 + 1) = 0$

$$(x_1 + \frac{1}{2}x_2)^2 + \frac{3}{4}x_2^2 + 1$$

$$\geq 1 \quad \forall x_1, x_2 \in \mathbb{R}.$$

Hence $x_1 = x_2$ and, by symmetry, $x_1 = x_2 = x_3.$

and

$$D^2 \phi(\underline{x}) = \begin{pmatrix} 3x_1^2 & -1 & -1 \\ -1 & 3x_2^2 & -1 \\ -1 & -1 & 3x_3^2 \end{pmatrix}.$$

Stationary points are $x_1 = x_2 = x_3 = \alpha$, where

$$0 = \alpha^3 - 2\alpha = \alpha(\alpha^2 - 2),$$

i.e. $\alpha = 0, \pm\sqrt{2}.$

Now

$$D^2 \phi(0) = \begin{pmatrix} 0 & -1 & -1 \\ -1 & 0 & -1 \\ -1 & -1 & 0 \end{pmatrix} = I - \underline{e}\underline{e}^T, \quad \underline{e} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}.$$

Hence $D^2 \phi(0) \cdot \underline{v} = \underline{v}$ if $\underline{v}^T \underline{e} = 0$

and $D^2 \phi(0) \cdot \underline{e} = \underline{e} - \underline{e}(\underline{e}^T \underline{e}) = -2\underline{e},$

so eigenvalues are 1 (twice), -2 & it's a saddle.

However, $\phi(\sqrt{2}, \sqrt{2}, \sqrt{2}) = -3 = \phi(-\sqrt{2}, -\sqrt{2}, -\sqrt{2})$

and

$$D^2 \phi(\sqrt{2}, \sqrt{2}, \sqrt{2}) = \begin{pmatrix} 6 & -1 & -1 \\ -1 & 6 & -1 \\ -1 & -1 & 6 \end{pmatrix} = 7I - \underline{e}\underline{e}^T$$

with eigenvalues 7 (twice) and 4 $\Rightarrow$ local minimum.

Now $f : \mathbb{R}^3 \rightarrow \mathbb{R}$ is infinitely differentiable & positive and unbounded for all suff large $\|\underline{x}\|$, so local min is global minimum.

Finally, near zero we have

$$0 = \frac{1}{2} \underline{x}^T D^2 \phi(0) \underline{x} + \dots$$

and, if $\underline{x} = \begin{pmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \\ 0 \end{pmatrix} u_1 + \begin{pmatrix} +1/\sqrt{2} \\ 0 \\ -1/\sqrt{2} \end{pmatrix} u_2 + \frac{1}{\sqrt{3}} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} u_3$

then

$$0 = u_1^2 + u_2^2 - 2u_3^2$$

i.e. for $r^2 = u_1^2 + u_2^2$ we have

$$\begin{aligned} 0 &= r^2 - 2u_3^2 \\ &= (r + \sqrt{2}u_3)(r - \sqrt{2}u_3) \end{aligned}$$

or $r = \pm \sqrt{2}u_3$,

i.e. semiangle $\tan^{-1}(\sqrt{2})$.

