

10.

$$L(y) = \frac{c_0}{\sqrt{1-ky}}, \quad y \in (0, h),$$

where $k \in (0, 1/h)$ (avoiding singularity).

Fermat $\Rightarrow$ find max/min of

$$\int_{-x_0}^{x_0} f(x, y, y') \, dx$$

where

$$f(x, y, y') = c_0^{-1} \sqrt{1-ky} \sqrt{1+(y')^2}, \quad \text{if } y \equiv y(x),$$

$$\text{Now } \frac{\partial f}{\partial x} = 0 \Rightarrow f - y' \frac{\partial f}{\partial y'} = A$$

i.e.

$$(1-ky)^{\frac{1}{2}} \left[(1+(y')^2)^{\frac{1}{2}} - (y')^2 (1+(y')^2)^{-\frac{1}{2}} \right] = A c_0$$

$$\Rightarrow (1-ky)^{\frac{1}{2}} \left[(1+(y')^2) - (y')^2 \right] = A c_0 (1+(y')^2)^{\frac{1}{2}}$$

$$\text{i.e.} \quad (1-ky)^{\frac{1}{2}} = A c_0 (1+(y')^2)^{\frac{1}{2}}$$

$$\Rightarrow 1-ky = A^2 c_0^2 (1+(y')^2)$$

$$\text{So } (y')^2 = A^{-2} c_0^{-2} (1-ky) - 1$$

$$\Rightarrow \frac{dy}{dx} = \left(A^{-2} c_0^{-2} (1-ky) - 1 \right)^{\frac{1}{2}}$$

$$\begin{aligned} \int dx &= \int \left(A^{-2} c_0^{-2} (1-ky) - 1 \right)^{\frac{1}{2}} dy \\ &= \int A c_0 (1 - A^2 c_0^2 - ky)^{\frac{1}{2}} dy \end{aligned}$$

$$x + \alpha = -\frac{1}{2} \frac{A c_0}{k} (1 - A^2 c_0^2 - ky)^{\frac{1}{2}}$$

$$\Rightarrow (x + \alpha)^2 = \left(\frac{1}{2} A c_0 k^{-1}\right)^2 (1 - A^2 c_0^2 - ky) \quad (1)$$

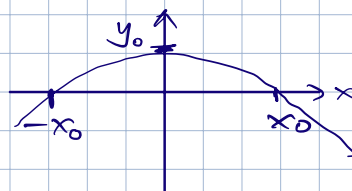
which is a parabola.

Now light ray enters medium at $(-x_0, 0)$,
leaves $(x_0, 0)$

and $y_0 = \max y$.

The quadratic has the equation

$$y = y_0 \left(1 - \frac{x^2}{x_0^2}\right).$$



Returning to

$$1 - ky = A^2 c_0^2 (1 + (y')^2).$$

we have $y' = -2xy_0/x_0^2$

$$1 - ky_0 \left(1 - \frac{x^2}{x_0^2}\right) = A^2 c_0^2 \left(1 + \frac{4x^2 y_0^2}{x_0^4}\right)$$

& equating coefficients gives

$$1 - ky_0 = A^2 c_0^2$$

$$\text{and } \frac{ky_0}{x_0^2} = A^2 c_0^2 \cdot \frac{4y_0^2}{x_0^4} \quad \text{OR } k = \frac{4A^2 c_0^2 y_0}{x_0^2}.$$

$$\text{So } 1 - ky_0 = A^2 c_0^2 = \frac{1}{4} \frac{kx_0^2}{y_0}$$

or

$$y_0(1 - ky_0) = \frac{1}{4} kx_0^2, \text{ i.e. } 4ky_0(1 - ky_0) = k^2 x_0^2.$$