

2 (i)

$$f(x, y) = x^2 y^{-1}, \quad x \in \mathbb{R}, y > 0$$

$$\nabla f = \begin{pmatrix} 2xy^{-1} \\ -x^2 y^{-2} \end{pmatrix}, \quad D^2 f(x, y) = \begin{pmatrix} 2y^{-1} & -2xy^{-2} \\ -2xy^{-2} & 2x^2 y^{-3} \end{pmatrix}$$

$$\begin{aligned} \text{or } D^2 f(x, y) &= 2y^{-1} \begin{pmatrix} 1 & -xy^{-1} \\ -xy^{-1} & x^2 y^{-2} \end{pmatrix} \\ &= 2y^{-1} \begin{pmatrix} 1 \\ -xy^{-1} \end{pmatrix} \begin{pmatrix} 1 \\ -xy^{-1} \end{pmatrix}^T. \end{aligned}$$

Thus

$$\underline{v}^T D^2 f(x, y) \underline{v} = 2y^{-1} \left[ \underline{v}^T \begin{pmatrix} 1 \\ -xy^{-1} \end{pmatrix} \right]^2 \geq 0$$

& we have convexity.

2 (i) First note that, for  $\underline{x} \neq \underline{x}'$  in  $H = \{(x, y) \in \mathbb{R}^2 : y > 0\}$  and for  $0 \leq t \leq 1$ , we have

$$\begin{aligned} & F(t\underline{x} + (1-t)\underline{x}') \\ &= F(t\underline{x} + (1-t)\underline{x}', \quad ty + (1-t)y') \\ &= (ty + (1-t)y') f\left(\frac{tx + (1-t)x'}{ty + (1-t)y'}\right). \end{aligned} \quad (1)$$

Now

$$\begin{aligned} & tF(\underline{x}) + (1-t)F(\underline{x}') \\ &= ty f\left(\frac{x}{y}\right) + (1-t)y' f\left(\frac{x'}{y'}\right) \\ &= [ty + (1-t)y'] \left\{ \frac{ty}{ty + (1-t)y'} f\left(\frac{x}{y}\right) + \frac{(1-t)y'}{ty + (1-t)y'} f\left(\frac{x'}{y'}\right) \right\} \\ &\quad \underbrace{\hspace{10em}}_{\text{CONVEXITY OF } f: \mathbb{R} \rightarrow \mathbb{R}} \\ &\geq f\left(\frac{ty \cdot \frac{x}{y} + (1-t)y' \cdot \frac{x'}{y'}}{ty + (1-t)y'}\right) \\ &= f\left(\frac{tx + (1-t)x'}{ty + (1-t)y'}\right). \end{aligned} \quad (2)$$

Comparing (1) & (2), we have convexity of  $F$  on  $H$ .

2 (ii) Here  $f: \mathbb{R} \rightarrow \mathbb{R}$  is univariate convex and the perspective  $F: \mathbb{R} \times \mathbb{R}_+ \rightarrow \mathbb{R}$  is defined by

$$F(x, y) = y f(x y^{-1})$$

[ If  $x = r \sin \theta$ ,  $y = r \cos \theta$ , then  $F(r, \theta) = r \cos \theta f(\tan \theta)$  for  $r \geq 0$  and  $\cos \theta > 0$ , i.e.  $|\theta| < \frac{\pi}{2}$ . ]

LINEAR ON EVERY RAY BUT not diff at (0,0). [ANALYSIS II]

Hessian:

$$F_x = y f'(x y^{-1}) \cdot y^{-1} = f'(x y^{-1})$$

$$F_y = f(x y^{-1}) + y f'(x y^{-1}) (-x y^{-2})$$

$$\text{or } F_y = f(x y^{-1}) - x y^{-1} f'(x y^{-1})$$

$$F_{xx} = f''(x y^{-1}) y^{-1}$$

$$F_{xy} = f''(x y^{-1}) (-x y^{-2})$$

CHECK:

$$F_{yx} = f'(x y^{-1}) y^{-1} + y f''(x y^{-1}) y^{-1} (-x y^{-2}) + y f'(x y^{-1}) (-y^{-2})$$

$$\text{or } F_{yx} = -x y^{-2} f'(x y^{-1}). \quad \checkmark$$

Finally,

$$F_{yy} = f'(x y^{-1}) (-x y^{-2}) + x y^{-2} f'(x y^{-1}) - x y^{-1} f''(x y^{-1}) (-x y^{-2})$$

$$\text{i.e. } F_{yy} = x^2 y^{-3} f''(x y^{-1}).$$

Thus

$$\begin{aligned} D^2 F(x, y) &= f''(xy^{-1}) \begin{pmatrix} y^{-1} & -xy^{-2} \\ -xy^{-2} & x^2y^{-3} \end{pmatrix} \\ &= f''(xy^{-1}) y^{-1} \begin{pmatrix} 1 & -xy^{-1} \\ -xy^{-1} & x^2y^{-2} \end{pmatrix} \\ &= f''(xy^{-1}) y^{-1} \begin{pmatrix} 1 \\ -xy^{-1} \end{pmatrix} \begin{pmatrix} 1 \\ -xy^{-1} \end{pmatrix}^T, \end{aligned}$$

so  $f$  convex,  $y > 0 \Rightarrow D^2 F(x, y)$  non-negative def  
& rank-one.

Now that

$$\begin{pmatrix} 1 \\ -xy^{-1} \end{pmatrix}^T \begin{pmatrix} xy^{-1} \\ 1 \end{pmatrix} = 0,$$

$\Rightarrow D^2 F(x, y)$  has eigenvectors  $(x = r \sin \theta, y = r \cos \theta)$

$$\begin{pmatrix} \cos \theta \\ -\sin \theta \end{pmatrix}$$

and

$$\begin{pmatrix} \sin \theta \\ \cos \theta \end{pmatrix}$$

[zero eigenvalue].