

$$5.(i) \quad A = [s(s-a)(s-b)(s-c)]^{1/2}, \quad s = \frac{1}{2}(a+b+c)$$

Want to max A s.t. $s = s_0$

i.e.

$$\max A(a, b, c)$$

$$\text{s.t. } g(a, b, c) = a + b + c = 2s$$

$$\text{Then } \nabla A = \lambda \nabla g \quad \&$$

$$\partial_a A = \frac{1}{2} A^{-1} \frac{\partial}{\partial a} [-s(a-s)(b-s)(c-s)]$$

$$= -\frac{1}{2} A^{-1} s(b-s)(c-s)$$

$$= -\frac{1}{2} A^{-1} \cdot s(a-s)(b-s)(c-s) \cdot (a-s)^{-1}$$

$$= -\frac{1}{2} A (a-s)^{-1}$$

$$\text{Hence } \nabla A = -\frac{1}{2} A \begin{pmatrix} (a-s)^{-1} \\ (b-s)^{-1} \\ (c-s)^{-1} \end{pmatrix}$$

$$\text{and } \nabla g = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

$$\text{So } -\frac{1}{2} A \cdot \frac{1}{a-s} = \lambda = -\frac{1}{2} A \cdot \frac{1}{b-s} = -\frac{1}{2} A \cdot \frac{1}{c-s}$$

$$\text{Thus } A > 0 \rightarrow s-a = s-b = s-c = 2\lambda/A$$

$$\text{or } a = s - 2\lambda A^{-1} = b = c.$$

5 (ii) Want largest area for right-angled triangle of given perimeter.

Let's start again $\Rightarrow$ ignore Heron's formula.

Instead, we

$$\begin{aligned} \max \quad & A = ab \\ \text{s.t.} \quad & \underbrace{a + b + \sqrt{a^2 + b^2}}_{g(a,b)} = p \equiv 2s \end{aligned}$$

$$\text{Then} \quad \nabla A = \begin{pmatrix} b \\ a \end{pmatrix} = \lambda \nabla g = \lambda \begin{pmatrix} 1 + \frac{1}{2}(a^2 + b^2)^{-\frac{1}{2}} \cdot 2a \\ 1 + \frac{1}{2}(a^2 + b^2)^{-\frac{1}{2}} \cdot 2b \end{pmatrix}$$

$$\text{or} \quad \begin{pmatrix} b \\ a \end{pmatrix} = \lambda \begin{pmatrix} 1 + \frac{a}{\sqrt{a^2 + b^2}} \\ 1 + \frac{b}{\sqrt{a^2 + b^2}} \end{pmatrix}.$$

Now $\sqrt{a^2 + b^2} = p - a - b$, so

$$\begin{aligned} \begin{pmatrix} b \\ a \end{pmatrix} &= \lambda \begin{pmatrix} 1 + \frac{a}{p-a-b} \\ 1 + \frac{b}{p-a-b} \end{pmatrix} \\ &= \lambda \begin{pmatrix} \frac{p-b}{p-a-b} \\ \frac{p-a}{p-a-b} \end{pmatrix}. \end{aligned}$$

$$\text{Then} \quad 1 = \lambda \frac{b^{-1}(p-b)}{p-a-b} = \lambda \frac{a^{-1}(p-a)}{p-a-b}$$

or

$$b^{-1}p - 1 = \bar{a}^{-1}p - 1$$

i.e. $a = b$ and $p = 2a + \sqrt{2a^2}$, area $= \frac{1}{2}a^2$.

Thus $p = (2 + \sqrt{2})a$, area $= \frac{1}{2} \cdot \frac{1}{(2 + \sqrt{2})^2}$.

5(ii) AGAIN: $a = r \cos t$, $b = r \sin t$, $\text{area} = \frac{1}{2} ab$,

perimeter $p = a + b + \sqrt{a^2 + b^2}$, i.e.

$$A(r, \theta) = \frac{1}{2} r^2 \sin t \cos t$$

&

$$p = r (\cos t + \sin t + 1).$$

So want to maximize

$$f(t) = \frac{1}{2} p^2 \frac{\sin t \cos t}{\cos t + \sin t + 1}.$$

Now

$$f'(t) = \frac{1}{2} p^2 \left\{ \frac{(c+s+1)(c^2-s^2) - sc(-s+c)}{(c+s+1)^2} \right\}$$

$$\begin{aligned} \{ \cdot \} &= (c+s+1)(c^2-s^2) + s^2c - sc^2 \\ &= (c+s+1-s)c^2 + s^2(c-c-s-1) \\ &= (c+1)c^2 - s^2(s+1) \\ &= c^3 + c^2 - s^3 - s^2 \\ &= (c-s)(c^2+sc+s^2) + (c-s)(c+s) \\ &= (c-s)(1+c+s+sc) \\ &= (c-s)(1+c)(1+s) \end{aligned}$$

Avoiding the degenerate cases $1+c=0$ and $1+s=0$,

$$\text{we have } c = s = \frac{1}{\sqrt{2}}$$