

6. We maximize $f(x, y, z) = xyz$
 subject $g(x, y, z) = \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1.$

OR

$$\max F(u, v, w) = abc \, uvw$$

$$\text{s.t. } G(u, v, w) = u^2 + v^2 + w^2 = 1.$$

Then $\nabla F = abc \begin{pmatrix} vw \\ wu \\ uv \end{pmatrix}, \quad \nabla G = \begin{pmatrix} 2u \\ 2v \\ 2w \end{pmatrix}$

& we have $\nabla F = \lambda \nabla G, \therefore \lambda = \frac{1}{2} (p = 2\lambda)$

$$abc \begin{pmatrix} vw \\ wu \\ uv \end{pmatrix} = p \begin{pmatrix} u \\ v \\ w \end{pmatrix}, \quad u^2 + v^2 + w^2 = 1.$$

Then $u = abc p^{-1} vw = (abc p^{-1})^3 wu \cdot uv$

or $1 = (abc p^{-1})^3 uvw.$

Hence $u^2 = abc p^{-1} uvw = (abc p^{-1})^{-2}$

and similarly for v^2, w^2 , which implies

$$1 = u^2 + v^2 + w^2 = 3 (abc p^{-1})^{-2}$$

Thus $u^2 = v^2 = w^2 = (abc p^{-1})^{-2} = \frac{1}{3}$

or $(u, v, w) = \left(\pm \frac{1}{\sqrt{3}}, \pm \frac{1}{\sqrt{3}}, \pm \frac{1}{\sqrt{3}} \right).$

Finally, $uvw = (abc p^{-1})^{-3} = \frac{1}{3^{3/2}} = \frac{1}{\sqrt{27}}.$