



$y = y(x)$ even, $y < 0$.

Energy conservation $\Rightarrow$

$$\frac{1}{2} (\dot{x}^2 + \dot{y}^2) + gy = 0,$$

Now $\dot{y} = \frac{dy}{dx} \cdot \dot{x} \equiv y' \dot{x}$

so that

$$\frac{1}{2} (1 + (y')^2) \cdot \dot{x}^2 = -gy$$

i.e.

$$\dot{x}^2 = \frac{-2gy}{1 + (y')^2}.$$

(RECALL)
 $y < 0$.

Travel time

$$T = \int_0^T dt = \int_0^T \frac{dt}{dx} dx$$

$$= \int_{-\frac{l}{2}}^{\frac{l}{2}} \frac{1}{\dot{x}} dx$$

$$T = \int_{-\frac{l}{2}}^{\frac{l}{2}} \underbrace{\sqrt{\frac{1 + (y')^2}{-2gy}}}_{F(x, y, y')} dx$$

Now $\frac{\partial F}{\partial x} = 0 \Rightarrow F - y' \frac{\partial F}{\partial y'} = C$

Then $\frac{\partial F}{\partial y'} = \cancel{\frac{1}{2}} \left(\frac{1+(y')^2}{-2gy} \right)^{-\frac{1}{2}} \cdot \cancel{2} \frac{y'}{-2gy}$

So

$$\begin{aligned} C &= \left(\frac{1+(y')^2}{-2gy} \right)^{-\frac{1}{2}} - \frac{(y')^2}{-2gy} \left(\frac{1+(y')^2}{-2gy} \right)^{-\frac{1}{2}} \\ &= \frac{\frac{1+(y')^2}{-2gy} - \frac{(y')^2}{-2gy}}{\sqrt{\frac{1+(y')^2}{-2gy}}} \\ &= \frac{1}{-2gy \sqrt{\frac{1+(y')^2}{-2gy}}} \end{aligned}$$

i.e. $C = \frac{1}{\sqrt{-2gy[1+(y')^2]}}$

Then $C^2 = \frac{1}{-2gy[1+(y')^2]}$

$$\approx 1+(y')^2 = \frac{-1}{2gyC^2}$$

$$\Rightarrow (y')^2 = -1 - \frac{1}{2gyC^2}$$

$$= - \left[\frac{1 + \frac{1}{2gyC^2}}{2gyC^2} \right]$$

1

TRICK: $y < 0$ and so need

$$1 + 2gy c^2 \geq 0$$

for otherwise $(y')^2 < 0$. Hence

$$2gy c^2 \geq -1$$

or $|y| \leq (2gc^2)^{-1}$.

Let $2gy c^2 = -\sin^2 \theta$.

Then (1) becomes

$$(y')^2 = \frac{1 - \sin^2 \theta}{\sin^2 \theta}$$

(2)

and

$$\frac{dy}{dx} = \frac{dy}{d\theta} \cdot \frac{d\theta}{dx} = \frac{dy}{d\theta} / \frac{dx}{d\theta}.$$

(3)

So (2) & (3) $\Rightarrow$

$$\frac{\left(\frac{dy}{d\theta}\right)^2}{\left(\frac{dx}{d\theta}\right)^2} = \frac{1 - \sin^2 \theta}{\sin^2 \theta} = \frac{\cos^2 \theta}{\sin^2 \theta}$$

(4)

and

$$2gy c^2 \frac{dy}{d\theta} = -2 \sin \theta \cos \theta$$

or

$$gc^2 \frac{dy}{d\theta} = -\sin \theta \cos \theta.$$

(5)

Substituting (5) in (4) yields

$$(gc^2)^{-2} \frac{\sin^2 \theta \cos^2 \theta}{\left(\frac{dx}{d\theta}\right)^2} = \frac{\cos^2 \theta}{\sin^2 \theta}$$

or $\left(\frac{dx}{d\theta}\right)^2 = (gc^2)^2 \sin^4 \theta$

i.e. $\frac{dx}{d\theta} = (gc^2) \sin^2 \theta$
 $= \frac{1}{2}(gc^2)(1 - \cos 2\theta)$

$$\Rightarrow \begin{cases} x(\theta) = \frac{1}{2}(gc^2)\left(\theta - \frac{1}{2}\sin 2\theta\right) + A, \\ y(\theta) = -(2gc^2)^{-1} \sin^2 \theta. \end{cases}$$

$y(0) = y(\pi) = 0 \Rightarrow x(0) = -l/2, x(\pi) = l/2,$
 so that $A = \frac{l}{2}$ and

$$\frac{l}{2} = \frac{1}{2} gc^2 \pi - \frac{l}{2}, \text{ or } l = gc^2 \pi.$$

Thus $c = \left(\frac{l}{g\pi}\right)^{1/2}$, i.e., $gc^2 = \frac{l}{\pi}$.

Note that $\frac{dx}{d\theta}(0) = 0 \Rightarrow y'$ infinite!

Then $\begin{cases} x(\theta) = \frac{1}{2} \frac{l}{\pi} \left(\theta - \frac{1}{2}\sin 2\theta\right) + \frac{l}{2} \\ y(\theta) = -\frac{\pi}{2l} \sin^2 \theta. \end{cases}$

Now

$$T = \int_{-\frac{l}{2}}^{\frac{l}{2}} \underbrace{\sqrt{\frac{1+(y')^2}{-2gy}}}_{dx} dx$$