

# Real Analysis 5.5: Complex Numbers

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You can download these slides and the lecture videos from my office server

<http://econ109.econ.bbk.ac.uk/brad/analysis/>

**Recommended books:** Lara Alcock (2014), “How to Think about Analysis”, Oxford University Press.

J. C. Burkill (1978), “A First Course in Mathematical Analysis”, Cambridge University Press.

## Definition

Let

$$\mathcal{C} = \left\{ \begin{pmatrix} a & -b \\ b & a \end{pmatrix} : a, b \in \mathbb{R} \right\}$$

and

$$\mathbb{I} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad \mathbb{I} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

Thus

$$\mathcal{C} = \{a\mathbb{I} + b\mathbb{I} : a, b \in \mathbb{R}\}$$

and  $\mathbb{I}$  is the matrix for clockwise rotation through  $\pi/2$  radians.

We call  $\mathcal{C}$  the set of **rotation-enlargement** matrices, for reasons that will become clear later.

## Theorem (1 and J)

*Every rotation–enlargement matrix can be uniquely written in the form*

$$a\mathbb{1} + b\mathbb{I}.$$

*Further, the relation*

$$\mathbb{I}^2 = -\mathbb{1}$$

*implies the multiplication rule*

$$(a_1\mathbb{1} + b_1\mathbb{I})(a_2\mathbb{1} + b_2\mathbb{I}) = (a_1a_2 - b_1b_2)\mathbb{1} + (a_1b_2 + b_1a_2)\mathbb{I}$$

*and we observe that  $ZW = WZ$  for any  $Z, W \in \mathcal{C}$  because  $\mathbb{1}$  commutes with every matrix.*

## Theorem

$\mathcal{C}$  contains a copy of  $\mathbb{R}$ :

$$\{a\mathbb{1} : a \in \mathbb{R}\}.$$

The subset

$$i\mathbb{M} = \{b\mathbb{I} : b \in \mathbb{R}\}$$

is the set of all real multiples of the rotation matrix  $\mathbb{I}$ .

## Example

If  $Z = a\mathbb{1} + b\mathbb{I}$ , then

$$Z(a\mathbb{1} - b\mathbb{I}) = (a^2 + b^2)\mathbb{1}.$$

## Definition

We write  $Z^* = a\mathbb{1} - b\mathbb{I}$ , so that

$$ZZ^* = Z^*Z = (a^2 + b^2)\mathbb{1}.$$

The **modulus** of  $Z$  is defined by  $|Z| = \sqrt{ZZ^*} = \sqrt{a^2 + b^2}$ .

Note that  $Z^*$  is also the matrix transpose of  $Z$ .

## Theorem

If  $Z = a\mathbb{1} + b\mathbb{I}$  and  $|Z| > 0$ , then the matrix  $Z$  is invertible and

$$Z^{-1} = \frac{Z^*}{|Z|}.$$

## Theorem

If  $Z = a\mathbb{1} + b\mathbb{I}$  and  $|Z| = 1$ , then  $Z^{-1} = Z^*$ , i.e.  $ZZ^* = \mathbb{1}$ .

## Exercise

The **norm**  $\|v\|$  of any vector

$$v \in \mathbb{R}^2 = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$$

is just its Euclidean length, i.e.  $\|v\| = \sqrt{v_1^2 + v_2^2}$ . Show that  $\|Zv\| = \|v\|$  when  $|Z| = 1$ .

**TRADITIONAL NOTATION:**  $a + ib \equiv a\mathbb{1} + b\mathbb{I}$ .

In geometrical terms,  $a\mathbb{1} + b\mathbb{I}$  is a rotation-enlargement matrix: the scale factor is  $\sqrt{a^2 + b^2}$  and the rotation angle  $\theta$  satisfies  $\tan \theta = b/a$ .