Statistical Learning Coursework

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NEW Due Date: March 4th, 2024.

1. Let A be any real $m \times n$ matrix, where $m \geq n$. Throughout this question $O(k)$ will denote the set of real $k \times k$ orthogonal matrices. You may assume standard properties of orthogonal matrices if clearly stated.

(i). Define the singular value decomposition.

3 pts

(ii). Define the Frobenius norm and inner product.

4 pts

(iii). Prove that $\|QA\|_F = \|AR\|_F = \|A\|_F$, for any $A \in \mathbb{R}^{m\times n}$, $Q \in O(m)$ amd $R \in O(n)$.

4 pts

(iv). Given any matrix $A \in \mathbb{R}^{n\times n}$, with singular value decomposition $A = USV^T$, prove that

$$\|A - Q\|_F^2 = \|S - W\|_F^2,$$

for any $Q \in O(n)$, where $W = U^T QV$. Hence show that

$$\|A - Q\|_F^2 = \sum_{k=1}^{n} \left(s_k^2 - 2s_k W_{kk} + 1\right),$$

where $s_1, \ldots, s_n$ are the singular values of A.

4 pts

(v). Hence, or otherwise, prove that the non-linear least squares Procrustes problem

$$\min_{Q\in O(n)} \|A - Q\|_F,$$

where $A \in \mathbb{R}^{n\times n}$, is solved by setting $Q = UV^T$. Briefly describe one application of this Procrustes problem.

5 pts

2. (i). Describe the k-means clustering algorithm.

8 pts

(ii). Describe the PageRank algorithm. You should describe a suitable iterative method for obtaining the PageRank stationary distribution, but need not prove convergence.

12 pts

3. Let $A \in \mathbb{R}^{m \times n}$, where $m \geq n$, have singular value decomposition $A = USV^T$, where $U = (\mathbf{u}_1 \ \mathbf{u}_2 \ \cdots \ \mathbf{u}_m)$, $V = (\mathbf{v}_1 \ \mathbf{v}_2 \ \cdots \ \mathbf{v}_n)$ and the singular values $s_1, \ldots, s_n$ are all positive.

   (i). Prove that
$$A = \sum_{k=1}^{n} s_k \mathbf{u}_k \mathbf{v}_k^T.$$

6 pts

   (ii). For $1 \leq r \leq n$, define
$$A_r = \sum_{k=1}^{r} s_k \mathbf{u}_k \mathbf{v}_k^T.$$

   Prove that $A_r \mathbf{w} = 0$ if $\mathbf{w} \in \text{span}\,\{\mathbf{v}_{r+1}, \ldots, \mathbf{v}_n\}$.

3 pts

   (iii). Prove that $A_r \mathbf{x} \in \text{span}\,\{\mathbf{u}_1, \ldots, \mathbf{u}_r\}$, for any $\mathbf{x} \in \mathbb{R}^n$.

3 pts

   (iv). Calculate $\|A - A_r\|_F^2$, for $1 \leq r \leq n$.

3 pts

   (v). Prove that $\|(A - A_r)\mathbf{x}\|_2 \leq s_{r+1} \|\mathbf{x}\|_2$, for $1 \leq r \leq n$.

5 pts