The Bike Problem

Note Title 08/11/2007

Suppose n cyclists randomly permute their bikes. What's the probability that at least one cyclist has the correct bike?

More formally, our sample space X consists of all possible permutations of the integers $1, 2, \cdots, n$. In other words,

$$X = \{ (i_1, i_2, \cdots, i_n) : i_1, \cdots, i_n \text{ distinct in } 1, 2, \cdots, n \}.$$

We assume each of these elements is assigned the same probability $1/n!$.

Further, define

$$A_k = \{ x \in X : i_k = k \}, \quad \text{for } k = 1, \cdots, n.$$

Thus A_k is the set of outcomes in which cyclist k has bike k. We want to calculate the probability

$$\mathbb{P}(A_1 \cup A_2 \cup \cdots \cup A_n).$$

<u>Example:</u> If $n = 3$, then the sample space is

$$X = \{ (1\ 2\ 3), (1\ 3\ 2), (2\ 1\ 3), (2\ 3\ 1), (3, 1, 2), (3, 2, 1) \}.$$

Then

$$A_1 = \{ (1\ 2\ 3), (1\ 3\ 2) \},$$
$$A_2 = \{ (1\ 2\ 3), (3\ 2\ 1) \},$$
$$A_3 = \{ (1\ 2\ 3), (2\ 1\ 3) \}.$$

Here $\mathbb{P}(A_1 \cup A_2 \cup A_3) = 4/6 = 2/3$.

The solution requires the inclusion-exclusion formula, whose proof I shall briefly defer:

$$\mathbb{P}(A_1 \cup A_2 \cup \cdots \cup A_n)$$

$$= \sum_{k=1}^{n} \mathbb{P}(A_k) - \sum_{k_1 < k_2} \mathbb{P}(A_{k_1} \cap A_{k_2})$$

$$+ \sum_{k_1 < k_2 < k_3} \mathbb{P}(A_{k_1} \cap A_{k_2} \cap A_{k_3})$$

$$- \sum_{k_1 < k_2 < k_3 < k_4} \mathbb{P}(A_{k_1} \cap A_{k_2} \cap A_{k_3} \cap A_{k_4})$$

$$+ \cdots$$

$$+ (-1)^{n-1} \mathbb{P}(A_1 \cap \cdots \cap A_n).$$

To use the formula, we need to calculate

$$\mathbb{P}(A_{k_1} \cap \cdots \cap A_{k_m}).$$

Now

$$\mathbb{P}(A_1 \cap \cdots \cap A_m) = \frac{(n-m)!}{n!}.$$

Hence

$$\mathbb{P}(A_{k_1} \cap \cdots \cap A_{k_m}) = \frac{(n-m)!}{n!} \qquad (\text{WHY?})$$

and

$$\sum_{k_1 < \cdots < k_m} \mathbb{P}(A_{k_1} \cap \cdots \cap A_{k_m}) = \binom{n}{m} \mathbb{P}(A_1 \cap \cdots \cap A_m)$$

$$= \binom{n}{m} \frac{(n-m)!}{n!}$$

$$= \frac{n!}{m!\,(n-m)!} \cdot \frac{(n-m)!}{n!} = \frac{1}{m!}$$

Hence the inclusion-exclusion formula implies

$$\mathbb{P}(A_1 \cup \cdots \cup A_n) = 1 - \frac{1}{2!} + \frac{1}{3!} - \cdots + \frac{(-1)^{n-1}}{n!}$$

$$\xrightarrow[(n \to \infty)]{} \sum_{k=1}^{n} \frac{(-1)^{k-1}}{k!}$$

Now $e^{-1} = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!}$. Hence

$$\lim_{n \to \infty} \mathbb{P}(A_1 \cup \cdots \cup A_n) = 1 - e^{-1}.$$

Proving the Inclusion-Exclusion Formula

For any subset A of the sample space X, the indicator function $I_A : X \longrightarrow \{0,1\}$ is defined by

$$I_A(x) = \begin{cases} 1 & \text{if } x \in A \\ 0 & \text{if } x \notin A. \end{cases}$$

Example For $\Omega = \{(1\ 2\ 3), (1\ 3\ 2), (2\ 1\ 3), (2\ 3\ 1), (3\ 1\ 2), (3\ 2\ 1)\}$

$I_{A_1}(1\ 2\ 3) = I_{A_1}(1\ 3\ 2) = 1$, but

$I_{A_1}(2\ 1\ 3) = I_{A_1}(2\ 3\ 1) = I_{A_1}(3\ 1\ 2) = I_{A_1}(3\ 2\ 1)$

$= 0.$

The indicator function has some crucial properties :

Firstly,

$$1 - I_A(x) = \begin{cases} 0 & \text{if} \quad x \in A \\ 1 & \text{if} \quad x \notin A. \end{cases}$$

In other words,

$$I_{A^c}(x) = 1 - I_A(x),$$

where A^c is the complement of the set A.

Further,

$$I_{A \cap B}(x) = I_A(x)\, I_B(x).$$

Finally, we use de Morgan's law:

$$(A_1 \cup \cdots \cup A_n)^c = A_1^c \cap A_2^c \cap \cdots \cap A_n^c.$$

[$x \in (A_1 \cup \cdots \cup A_n)^c$ iff ($x \notin A_1$ and $\cdots$ and $x \notin A_n$).]

Thus

$$I_{A_1 \cup \cdots A_n}(x) = 1 - I_{(A_1 \cup \cdots \cup A_n)^c}(x)$$

$$= 1 - (1 - I_{A_1}(x))(1 - I_{A_2}(x)) \cdots (1 - I_{A_n}(x)).$$

This is known as the INCLUSION–EXCLUSION law.